Dynamical Systems with Controlled Singularities: Physically-Based Representation and Control-Oriented Modeling

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Abstract

A new class of systems: dynamical systems with controlled singularities, is introduced. This class refers to systems that admit impulsive control actions within their singular motion phases, the latter including changes in dimension, discontinuities in the state, and other nonsmooth types of motion. The motivation for this effort comes from various applications that admit sensing and/or actuation ultra-fast in comparison to the natural system time scale, such as systems and mechanisms with impact-induced motion, power networks under faults, fast positioning devices, smart skins, and switching electronic circuits. The present work focuses on the class of dynamical systems with singular motions arising due to system interaction with controlled, or active, state constraints. The latter are assumed to be parametrizable by the elasticity-like coefficient as follows: for the finite values of the coefficient the constraints are assumed to admit small violation dependent on the coefficient value, i.e., to display the elastic-type properties, whereas in the limit as this coefficient tends to infinity the constraint violation becomes inadmissible, i.e., the constraints become rigid. A physically well justified representation of the above-mentioned class of systems is developed. This representation is found to require equations with the unbounded impulsive control signals in the right-hand-side (rhs) to describe the singular phase, making it not well suited for controller synthesis. Using a specially constructed topological map parametrized by the coefficient of elasticity, with the latter tending to infinity, the original representation is shown to reduce in the limit to the simplified one given by the nonlinear generalized differential equations containing delta-functions in their rhs. This reduced, or limit, representation is shown to be a regularization of the original one due to the boundedness of both the control signals and the coefficients of the delta-functions in the rhs of the system equations, and, consequently, to possess compatibility with the regular control design techniques. It is also shown to be capable of generating a unique isolated discontinuous system motion, i.e., to provide a tight and well-behaved description of the collision with rigid constraint. It is then demonstrated that the corresponding paths generated by the original and the limit representations can be made arbitrarily close to each other uniformly except, possibly, in the vicinities of the jump points by the appropriate choice of the value of the elasticity coefficient. This is shown to permit enforcing, for sufficiently large values of the elasticity coefficient, of the desired limit system behavior onto the original system by simply taking the control signals found through the limit representation, time-rescaling them, and inserting the resulting signals directly into the original system. These features show that limit system, in fact, provides a control-oriented representation of the class of systems considered. Using this framework, two detailed examples of modeling in a difficult to model class of systems - mechanical systems with control actions introduced through collisions with controlled nonstationary constraints, are given.

Keywords

Nonsmooth analysis and synthesis, control of mechanical systems, singularities, systems with unilateral constraints, impacts, impulsive control, generalized solutions

I. Introduction

Systems that exhibit singularities in their behavior, such as discontinuities and nonsmoothness in system motion, jumps in dimension, lack of continuous dependence on the initial conditions, and nonuniqueness of solutions of the equations of motion are becoming more and more technologically important, and interest in their modeling and control, as well as development of new ultra-fast sensing and actuation capabilities during the singular phase is increasing ([14], [15], [22], [53], [58], [2], [3], [49]). For example, in mechanical systems collisions between the interacting bodies cause the abrupt change in their velocities and thereby create the discrete-continuous behavior. Specific examples include robotic manipulators [42], [51],[53], vibro-impact mechanisms [1], walking (and, potentially, jumping) biped robots [22], and many others (cf. [14] with its extensive bibliography). Moreover, in the cases of juggling mechanical systems [13] and micro-electromechanical systems (MEMS) with micro-actuator arrays and thin-film impact microactuators ([6]-[8], [25], and [31]) motion is induced only through organized impacts, hence it could be controlled only through these impacts. Such motion is especially important in MEMS since it is characterized by significantly reduced friction and adhesion, known to be severe problems in this area. Therefore, the effects of collision in modeling and control of these systems play a principal role and can not be neglected.

Another large area of systems with nonsmooth motions is power systems, both at the generation and distribution levels. In these systems faults often have devastating effects. The topology of the power network is abruptly impacted by the fault, thereby inducing nonsmoothness in the network dynamics and a possibility of further fault propagation

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[27], [54]. Other examples include space vehicles with impulsive propulsion [29], [30], economic systems with abrupt inputs in the form of sharp interest rate raises or cuts [18], [21], and quantum electronic systems [21].

Traditionally, the control e¤ort in such systems has been either exerted during the nonsingular phase of the system motion or spent to induce the a priori rigidly speci…ed singular phase that does not admit any corrective actions within it. For example, at present, in the case of singularities induced by collision with constraints ([24], [15], [53], [22]), the latter are viewed as "passive impacters" which, during the phase of the engagement of the system with the constraint, enter passively into the determination of the post-impact system motion and do not substantially alter the motions induced by the interaction from impact to impact.

Upon detailed examination, however, of impact games such as soccer, ping-pong, tennis, and volleyball one can notice that the advanced player looks beyond the "novice" objective of simply changing the trajectory of the ball (and thereby introducing the trajectory nonsmoothness and velocity discontinuity) and attempts to gain advantage by pursuing more sophisticated performance objectives, such as providing rotation of the ball and changing the velocity both in magnitude and direction, all during the short phase of interaction with the ball. The player in this setting can be viewed as generating a constraint and actively controlling its properties during the engagement phase. This player action gives rise to a new concept of active, or controlled, constraints, either naturally present or created through actuation, capable of radically changing the attainability set of the post-impact system state. The engagement phase of the system with such constraint can then be termed active singularity. These concepts, initially proposed by the authors in [3], [40], [41],[4], and [5], then naturally lead to the introduction of a new class of systems, dynamical systems with active, or controlled, singularities: a class of systems that admit control actions during the singular phases of their motion.

One immediately …nds a number of applications where these concepts could be very bene…cial or even critically important. For example, in cutting systems, at the end of each pass the cutting torch re-enters the cutting path through the oval-shaped turning phase with the same velocity in the opposite direction to assure the uniformity of the cutting depth. This results both in the extra scrap metal and the increased cutting cycle duration. It, therefore, looks attractive to simply reverse the direction of the torch at the cutting path endpoint. However, to ensure that the velocity-dependent quantity of heat delivered by the torch provides the required uniformity of cutting in the vicinity of the endpoints, the position- and velocity-dependent impulsive control action needs to be exerted on the motion of the torch at the turning point to maintain the velocity of cutting but instantly change its direction. Due to the impulsive forces required to execute such change, the dynamics of the turning phase will di¤er from that of the regular one and will require …ne structuring of the control signal for optimal execution of turning. Similar examples are positioning systems with precision/agility performance objectives, such as optical scanners, where it is necessary to apply magnetic/electric impulses to the actuators during the moment of the direction change to attain optimal performance and/or uniformity of exposure. Another example is fault propagation stopping in the power networks via fast fault clearing which could be plausibly implemented through the impulsive control actions exerted in a power system during the fault.

The present work focuses on the class of dynamical systems that encompasses most of the applications described above, namely, systems with controlled, or active, unilateral constraints. The latter are assumed to be parametrized by the coe¢ cient of elasticity: elastic, i.e. admitting a small constraint violation, for the …nite coe¢ cient value and rigid, i.e. forbidding the constraint violation, in the limit as the coe¢ cient tends to in…nity. Naturally, bringing out and full utilization of the capabilities added by active constraints call for modeling framework that can i) incorporate the relevant phenomenological details, and ii) conveniently lend itself to the overall motion planning and controller synthesis for both regular and singular phases to enforce this motion. However, satisfying these, often con‡icting, requirements for the class of systems considered turns out to be highly nontrivial. Indeed, examining available modeling techniques, for example, for mechanical systems subject to unilateral constraints with respect to the …rst requirement, one …nds that the conventional modeling of collisions, known to typically result in the velocity jumps in the system motion, makes use of either the so-called collision mapping or the restitution law, both of which give an expression for the velocity after the impact in terms of the velocity and position before the impact [1], [14], [15], [51]. The collision mapping, however, cannot analytically support description of the boundary of a time-varying and controlled, or active, constraint when the control forces during the phase of contact are impulsive.

The second requirement could be addressed by approximating the fast almost discontinuous motion of the actual system in the singular phase by the considerably simpler completely discontinuous, or limit, motion and, on the basis of the phenomenologically accurate model, generating the corresponding tight control-oriented models of limit dynamics with enough regularity to admit controller synthesis in both phases. Furthermore, the implications of applying the results obtained on the basis of such limit description to the actual system should be clari…ed, as well. Examining available techniques for modeling of limit behavior of systems with singular motions, one …nds that this motion has been traditionally rigorously described by quasi-di¤erential equations [49] and di¤erential inclusions [50], [44], [46]. These representations, however, give system evolution not in terms of an isolated trajectory, but a set of trajectories, referred to as integral funnel. As a result, system motion is vaguely de…ned and its precise computation and control are problematic. In [45], [46], and [26] it is indicated that the extraction of an isolated system trajectory from the funnel can be accomplished through the use of di¤erential equations with measure; however, the approach provided is not extendable to controlled singularities.

The present work addresses the issues raised above by making the following contributions to the existing literature:

- a novel conceptual framework, that of systems with controlled singularities, is proposed and used to introduce the class of dynamical systems with controlled, or active, elastic unilateral constraints;
- for this class of systems, a rigorous analytical framework is developed that resolves the aforementioned difficulties, specifically:
  - a system of differential equations with unbounded impulsive control signals in the right-hand-side (rhs), further referred to as the original system, that consistently accommodates controlled collisions, but is not well suited for motion planning and controller synthesis, is introduced;
  - a topological map - a one-to-one space-time transformation parametrized by the coefficient of elasticity is found that permits regularization of the original system;
  - applying this map to the original system, the corresponding limit representation is obtained for the infinite value of the elasticity coefficient, and it is demonstrated that the corresponding solutions of the original and the limit representations can be made arbitrarily close to each other uniformly except, possibly, in the vicinities of the jump points by the appropriate setting of the value of the elasticity coefficient in the original system;
  - a new type of description of the impulsive action in the form of the controlled shift operator along the trajectories of the equation of fast dynamics in the singular phase is introduced and incorporated into the system limit representation; this operator can be viewed as active collision mapping, replacing the traditional one;
  - the limit representation is shown to have the desired modeling properties, namely, to uniquely describe isolated paths with discontinuities, have bounded rhs delta-function coefficients and control signals, retain all the details of the original system dynamics, and, therefore, conveniently lend itself to simulation, system design, and control law synthesis;
  - implementation of the control laws is shown to be accomplished by simply substituting the time-rescaled bounded control signals found through the use of the limit representation into the actual physical system, resulting in the behavior of the latter close to that of the limit system for sufficiently large values of the elasticity coefficient and sufficiently accurate original system description;
- using this framework, two detailed examples of modeling in a difficult to model class of systems - mechanical systems with control actions introduced through collisions with controlled nonstationary constraints, are given.

The structure of the paper is as follows. Section II gives two motivating examples of systems that do not lend themselves to obtaining good models using the existing theory. The general model of a system with active constraints, the space-time transformation, and the multi-scale system description are introduced in Section III. The limit representations of systems with one singularity and multiple singularities are derived in Sections IV and V, respectively. Section VI takes the examples of Section II and shows in detail how to use the framework developed in Sections III - V to obtain their limit representation. Finally, Section VII draws the conclusions. Appendix provides the proofs of some technical results.

II. Motivating examples

In this section two examples of a racket interacting with a ball are introduced. Both examples do not admit tight description of their dynamics using the available modeling techniques. The difference between the examples is that in the first one the velocity of the boundary surface during an impact does not change abruptly, whereas in the second one the rate of the velocity change during an impact is of the same order as that of the velocity of a bouncing ball. The examples, therefore, represent systems with passive and active singularities, respectively. The constraints are assumed to be parametrized by the coefficient of elasticity μ, elastic and rigid for finite and infinite μ, respectively. The elasticity is permitted to be non-ideal, in general, with the non-ideality characterized by the restitution coefficient k.

A. Example 1: System with a Passive Singularity: Modeling of a Ball Colliding with Stationary and Moving Rackets

Consider a ball of mass m with a two-coordinate motion colliding with an elastic obstacle of mass M, such as a racket, that has a single coordinate motion. Let $(x_p^1; x_p^2)$ be the horizontal and the vertical coordinates of the moving ball, respectively, $(x_v^1; x_v^2)$ be the corresponding velocities, and $(X_p; X_v)$ be the coordinate and velocity of the obstacle surface in the vertical direction.

Then, the system has the state vector $z = (x_p^1; x_p^2; X_p; x_v^1; x_v^2; X_v)$. The constraint which defines the free motion area is given by the relation $G(z) = X_p - x_p^2 \quad 0$: The impact occurs at $t = \tau$ when $X_p(\tau) - x_p^2(\tau) = 0$.

The continuous motion description in the presence of the finite elasticity $\mu < \infty$ is standard. The goal at hand, however, is to obtain the velocity jump representation corresponding to the limit motion in the case of $\mu \to \infty$. In the absence of dry friction, this problem can be solved through the use of the mechanical conservation laws. In the presence of the dry friction, however, this problem becomes much more complicated, since both the elastic and the friction forces are of the impulsive type and their structures depend on the current abruptly changing vertical component of velocity. Figures 1 - 3 show the typical cases arising in this problem.

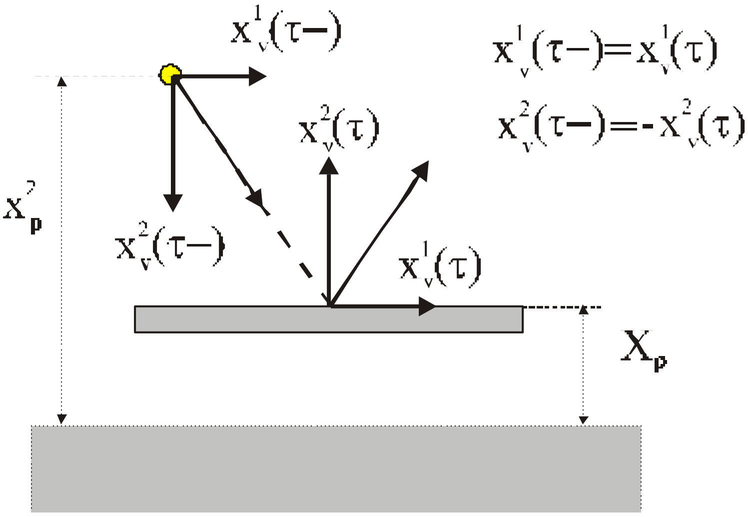

Fig. 1. Impact with stationary ideally-elactic surface, restitution coefficient k=1, no dry friction.

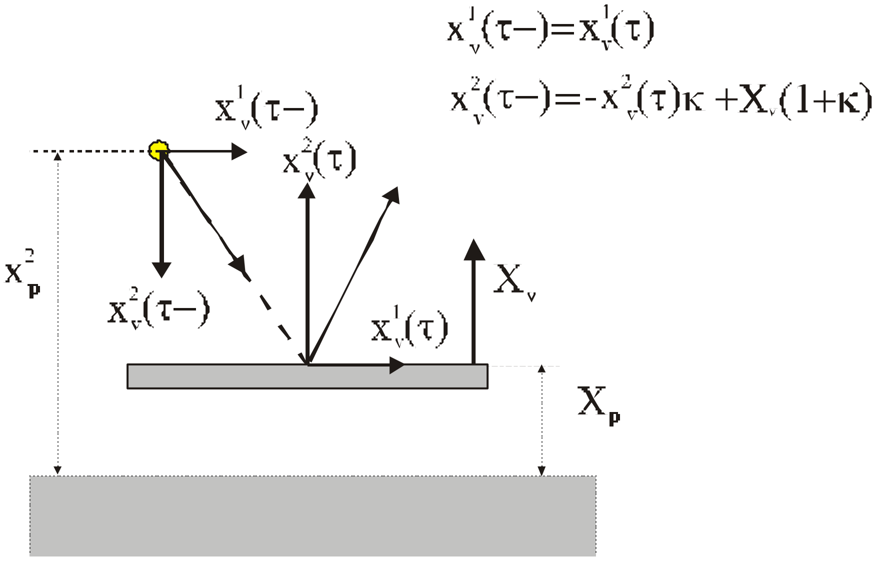

Fig. 2. Impact with moving non-ideally elastic surface, restitution coefficient ($0 < k < 1$), and no dry friction.

In the first case, with the presence of the ideal elasticity, the stationary racket, and the absence of dry friction, the velocity after the impact can be calculated as shown in Fig.1. For the sake of simplicity it is assumed that $m \ll M$ (the general case will be considered below in the Section VI).

The presence of the non-ideal elasticity and the movement of the racket modify the vertical velocity component of the ball as shown in Fig. 2

The impact in the presence of dry friction is shown in Fig. 3.

In the latter case it is not possible to calculate the horizontal velocity component of the ball solely on the basis of the conservation laws without taking into account the dynamics of the contact. Indeed, here the component $x_v^1(\tau)$ depends not only on all the velocities prior to the impact, but also on the profile of the elastic reaction force. Therefore, if one would like to obtain a tight description of the real motion, the impulsive nature of the impact has to be brought out in full detail. Otherwise, the only possibility would be to provide a rather loose system description through the use of differential inclusions like $x_v^1(\tau) < x_v^1(\tau-)$ or their analogs.

B. Example 2: System with an Active Singularity: Modeling of a Ball Colliding with a Moving Racket that Rotates During an Impact

Let us now consider the example motivated by the impact games, like ping-pong, where the player rotates the racket surface during the phase of contact of a racket with a ball. In this case, the additional force component arises due to the

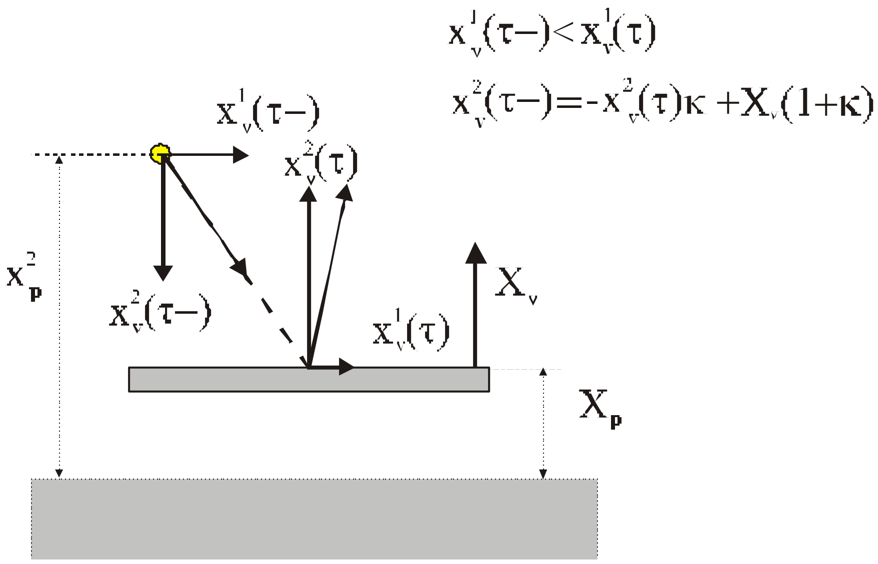

Fig. 3. Impact with moving non-ideally elastic surface, restitution coe¢cient (0 < k <1), and dry friction present.

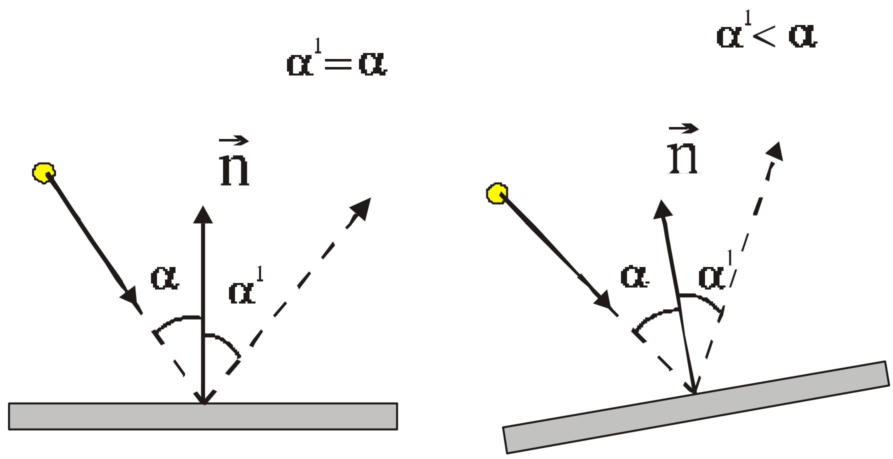

Fig. 4. Impact with fast rotating racket.

change of the racket surface orientation, as shown in Fig. 4, where ® and ®^{0} denote the pre- and post-impact re‡ection angles of the ball with respect to normals to the racket surface.

Such racket rotation during the very fast contact phase is frequently demonstrated by the highly skilled ping-pong players. If one tries to derive a collision map through the use of the mechanical conservation laws, one easily comes to the conclusion that these laws are insu¢cient for describing the impulsive reaction of the rotating elastic surface, since both the orientation and the value of the reaction force abruptly change during the impact phase. This situation presents an even greater modeling challenge than that in the previous example due to the abrupt change of several position coordinates (like the normal to the racket surface) during contact. The typical behavior of x_p(¢); x_v(¢) is shown in Fig. 5

The remainder of the present work develops the analytical setting capable of addressing the challenge posed by the examples described in this section.

III. Dynamical Systems with Controlled Singularities: the Physically-Based Model and the Multi-Scale Representation.

In order to approach the problems posed by the examples in Section II, one …rst needs to focus on the phase of the constraint engagement. In the collision of rigid bodies, the constraints are not perfectly rigid and undergo small violation, giving rise to the contact forces. Therefore, in reality, there is a very fast, but continuous, phase of motion that looks discontinuous only with respect to the velocities in the natural time scale. If one considers this phase in the enlarged spatio-temporal scale generated via some space-time transformation, one can obtain a more detailed description of the collision. In [33], [34], and [42] this approach is used to derive the equations of collision mapping for robotic manipulator;

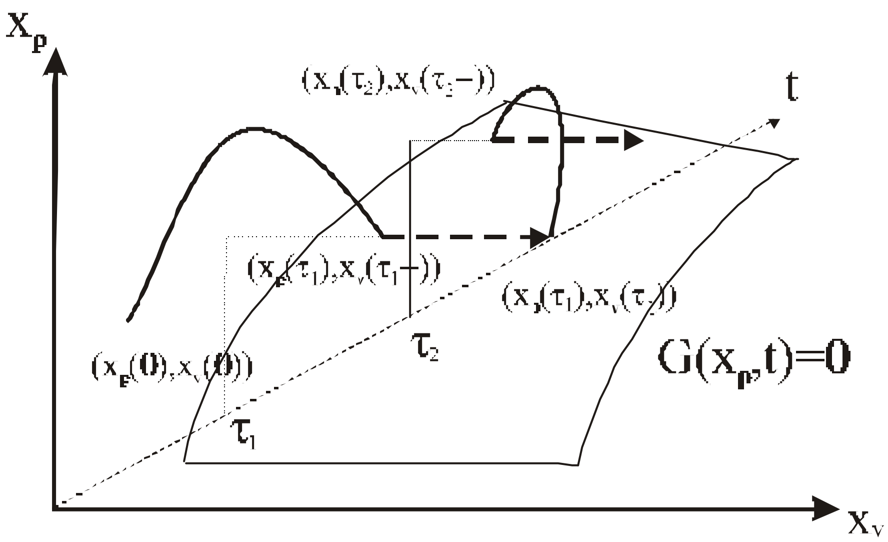

Fig. 5. Evolution of the limit system.

however, the description of the discontinuous system behavior is not rigorously obtained.

Further on, unlike the "passive constraints" case considered in [33], [34], and [42], in the case of the active constraints the impulsive control action should be admitted during the contact and properly represented in the system model. It looks reasonable to explore whether the formalism of the impulsive control theory (cf. [55], [56], [9], [10], [11], [12], [36], [37], [48] and many others) could be utilized for this purpose; however, one finds it to be not directly applicable to the case of constrained dynamical systems. This is due to the fact that the impulsive control setting considers large short duration control inputs to be external to the system, whereas large short duration contact forces arising during collision are inherently present in the constrained system itself and should be considered as the internal ones.

Based on these considerations, a physically motivated approach to generating the descriptions of systems with the contact impulsive forces is proposed with the following features:

- the contact force is considered to be the result of a small violation of a constraint that starts taking place as soon as the system hits the boundary of a constraint;
- this force resists the penetration of the system into the domain, inhibited by the constraint, and causes a very fast (almost instantaneous) change of the sign of normal velocity component to the opposite one;
- the controlled motion in the inhibited domain is admitted and described by the nonstandard controlled singularly perturbed differential equation with infinitely growing right-hand-side, such that the solution of this equation behaves in the limit as a stepwise function with respect to the components of the generalized velocity.

A. Natural and Singular Motion Phases for Systems with Controlled Singularities: the General System Model

Let in a system with elastic constraints the elasticity be parametrized by some coefficient μ. Let the constraints admit a system motion, albeit inhibited, within the area occupied by them for finite μ and become rigid for $\mu \to \infty$. Further on, consider decomposition of the entire system motion into two phases, the one corresponding to the motion in the area free of constraints and the second one describing the motion in the area inhibited by constraints, referred to as the natural and the singular phases, respectively.

Let the controlled dynamical system be described by the set of variables $x_p(t) \in R^n; x_v(t) \in R^n$; where vector x_p is referred to as the set of generalized coordinates and x_v as the set of generalized velocities. Suppose that there is some constraint given in the form of the inequality

$$G(x_p(t);t) \geq 0 \qquad (1)$$

were $G(x;t)$ is the continuous and sufficiently smooth function. Let the system admit the two modes of motion, an unconstrained, or motion in the constraint-free area, and an "inhibited", or motion in the area occupied by the constraint, further referred to as the natural and the singular motion phases, respectively.

A.1 Motion in the Natural Phase

In the domain $\{(x_p;t) : G(x_p;t) \le 0\}$ the system of differential equations for $\{x_p; x_v\}$ has the form

$$
\begin{aligned}
\dot{x}_p(t) &= F_p^r(x_p(t); x_v(t); u(t); t);\\
\dot{x}_v(t) &= F_v^r(x_p(t); x_v(t); u(t); t);
\end{aligned}
\tag{2}
$$

where $u(t) \in U \subset R^m$ is some control, U is a compact set, and $F_v^r(x_p; x_v; u; t)$ and $F_p^r(x_p; x_v; u; t)$ have the standard properties of continuity and smoothness sufficient for existence and uniqueness of the solution of system (2) for a given measurable control $u(\cdot)$ and arbitrary initial conditions $x_p(0); x_v(0)$. For example, they could be assumed to be continuous with respect to all variables and smooth with respect to $(x_p; x_v)$:

A.2 Motion in the Singular Phase

In the domain $\{(x_p;t) : G(x_p;t) > 0\}$ the system of differential equations for $\{x_p; x_v\}$ has the form

$$
\begin{aligned}
\dot{x}_p(t) &= F_p^r(x_p(t); x_v(t); u(t); t);\\
\dot{x}_v(t) &= \mu F_v^s(x_p(t); x_v(t); w(t;\mu); t; \mu) + F_v^r(x_p(t); x_v(t); u(t); t);
\end{aligned}
\tag{3}
$$

where $\mu F_v^s(x_p; x_v; w(t;\mu); t; \mu)$ describes an additional controlled contact force, with $w(t;\mu) \in W$ being the external contact force control signal. As before, this contact force is considered to arise due to the constraints violation. Function $F_v^s(x_p; x_v; w; t; \mu)$ is supposed to be continuous and smooth in the area $G(x_p; \tau) \ge 0$; and satisfy the constraints

$$
F_v^s(x_p; x_v; w; t; \mu) = 0; \quad \text{if} \quad G(x_p; t) = 0; \quad \left.\frac{d}{dt}\right|_{F_p^r} G(x_p; t) = 0:
$$

Assume that for any given $0 \le \mu < 1$ the joint system (2),(3) has the unique solution for any given measurable controls $u(\cdot); w(\cdot)$: The objective is to determine the behavior of the joint system for $\mu \uparrow 1$; and to find out if there exists the appropriate limit for its solution. If the limit exists, one can treat it as the generalized solution of a dynamical system with unilateral constraints, which would then be described by the limiting form of the joint system - a differential equation with delta functions in the rhs, or, more generally, with measure.

The equations (2) and (3) together with the constraint (1) will be referred to as the original system.

Remark 1: The principal feature of the systems considered is that their generalized coordinates are continuous while their velocities admit the jumps. At the same time, the generalized coordinates are exactly the ones responsible for the appearance of the contact forces. Therefore, in order to properly describe system behavior one needs to carry out a more involved analysis in the vicinity of the constraint violation points. It should be noted that the standard machinery of the singular perturbations analysis does not provide the possibility for clearly bringing out the nature of the impacts. The typical results that could be obtained with the aid of the singular perturbations analysis give the solutions that do not depend on the state preceding the impact phase. This, however, can not be the case in impact dynamics. Therefore, to describe the genuine impulsive nature of the impact, one needs to introduce the multi-scale motion representation obtained through "opening up" of a singularity, i.e. modeling of its fine structure, which can be accomplished through the use of the space-time transformation in the vicinity of the singularity point.

B. Space-Time Transformation at the Singularity Point and the Multi-Scale System Description

Let the system start from the initial condition $x_p(0); x_v(0)$ such that $G(x_p(0); 0) < 0$ and τ be the first point where the system engages the constraint, so that

$$
G(x_p(\tau); \tau) = 0; \quad \left.\frac{d}{dt}\right|_{F_p^r} G(x_p(\tau); \tau)) > 0; \tag{4}
$$

and control $w(t;\mu)$ has a form

$$
w(t;\mu) = \begin{cases} w_1\left(\dfrac{t-\tau}{1-\mu^{1=2}}\right); & \text{if} \quad t \ge \tau;\\ 0; & \text{otherwise}: \end{cases} \tag{5}
$$

Therefore, for finite value of μ there exists a non-zero time interval of the constraints violation. Introduce the following space-time transformation of the coordinates and the independent variable for $s > 0$:

$$
\begin{aligned}
y_p^{\mu}(s) &= x_p(\tau) + \mu^{1/2}[x_p(\tau + \mu^{-1/2}s) - x_p(\tau)];\\
y_v^{\mu}(s) &= x_v(\tau + \mu^{-1/2}s);\\
t &= \tau + \mu^{-1/2}s.
\end{aligned}
\tag{6}
$$

Then the new variables $\{y_p^{\mu}(s); y_v^{\mu}(s)\}$ satisfy the equations

$$
\begin{aligned}
y_p^{\mu}(s) &= F_p^r\left(\frac{y_p^{\mu}(s) - x_p(\tau)}{\mu^{1/2}} + x_p(\tau); y_v^{\mu}(s); u(\tau + \mu^{-1/2}s); \tau + \mu^{-1/2}s\right);\\
y_v^{\mu}(s) &= \mu^{1/2}F_v^s\left(\frac{y_p^{\mu}(s) - x_p(\tau)}{\mu^{1/2}} + x_p(\tau); y_v^{\mu}(s); w_1(s); \tau + \mu^{-1/2}s; \mu\right) +\\
&\quad \mu^{-1/2}F_v^r\left(\frac{y_p^{\mu}(s) - x_p(\tau)}{\mu^{1/2}} + x_p(\tau); y_v^{\mu}(s); u(\tau + \mu^{-1/2}s); \tau + \mu^{-1/2}s\right);\\
y_p^{\mu}(0) &= x_p(\tau); \quad y_v^{\mu}(0) = x_v(\tau -).
\end{aligned}
\tag{7}
$$

The system of equations (2) for the nonsingular phase, the coordinate map (6), and the system of equations (7) for the singular phase will be jointly referred to as the multi-scale motion representation of the original system. This term arises due to the decomposition of the original system equations (2) and (3) that contain mixed scales into subsystems separately describing the slow regular and the fast singular phases.

IV. Dynamical Systems with Controlled Singularities: Limit Representation of a Single Jump.

Next theorem describes the limit behavior of singularly perturbed system (7) as $\mu \uparrow \infty$. The theorem and its corollary demonstrate that the velocity jumps can be represented by means of the shift-operator along the paths of some limit system of differential equations. The theorem also shows how to incorporate control into the singular motion phase and thereby creates the bridge between impact mechanics and the impulsive control theory.

A. Calculation of the jump of the generalized velocity

Assumption 1: Suppose that F_v^s satisfies the Lipschitz condition in the following form: there exists $L > 0, \mu_0 > 0$ such that for any (x_p, x_p', x_v, x_v'), $t \in [0, T]$, $w \in W$, and $\mu \geq \mu_0$

$$
\|F_v^s(x_p, x_v, w, t, \mu) - F_v^s(x_p', x_v', w, t, \mu)\| \leq L\{\|x_p - x_p'\| + \mu^{-1/2}\|x_v - x_v'\|\}.
\tag{8}
$$

Theorem 1: Assume that:

1. 1) for any (x_p, τ) such that $G(x_p, \tau) = 0$ there exists

$$
\begin{aligned}
&\lim_{\mu \uparrow \infty} F_p^r\left(\frac{\tilde{y}_p - x_p}{\mu^{1/2}} + x_p; \tilde{y}_v; u(\tau + \mu^{-1/2}s); \tau + \mu^{-1/2}s\right) =\\
&= \tilde{F}_p(\tilde{y}_p; \tilde{y}_v; u(\tau); x_p; \tau);\\
&\lim_{\mu \uparrow \infty} \mu^{1/2}F^s\left(\frac{\tilde{y}_p - x_p}{\mu^{1/2}} + x_p; \tilde{y}_v; w_1(s); \tau + \mu^{-1/2}s; \mu\right) =\\
&\tilde{F}_v(\tilde{y}_p; \tilde{y}_v; s; w_1(s); x_p; \tau);
\end{aligned}
\tag{9}
$$

where convergence is uniform in any bounded vicinity of $(\tilde{y}_p, \tilde{y}_v, s)$;
2) the limit system of differential equations, i.e.

$$
\begin{aligned}
\tilde{y}_p(s) &= \tilde{F}_p(\tilde{y}_p(s); \tilde{y}_v(s); w_1(s); x_p(\tau); \tau);\\
\tilde{y}_v(s) &= \tilde{F}_v(\tilde{y}_p(s); \tilde{y}_v(s); s; w_1(s); x_p(\tau); \tau);\\
\tilde{y}_p(0) &= x_p(\tau); \quad \tilde{y}_v(0) = x_v(\tau -).
\end{aligned}
\tag{10}
$$

has the unique solution on some interval $[0; s^* + \varepsilon]$; where $\varepsilon > 0$ and

$$s^* = \inf \left\{ s > 0 : \begin{array}{l} G'_t\Big|_{(x_p(\tau);\tau)} s + G'_x\Big|_{(x_p(\tau);\tau)} (\bar y_p(s) - x_p(\tau)) = 0; \\ G'_t\Big|_{(x_p(\tau);\tau)} + G'_x\Big|_{(x_p(\tau);\tau)} \bar F_p(\bar y_p(s); \bar y_v(s); u(\tau); x_p(\tau); \tau) < 0 \end{array} \right\}: \qquad (11)$$

Then, if $\nu \to \infty$;

$$(y^\nu_p(s); y^\nu_v(s)) \to (\bar y_p(s); \bar y_v(s)) \quad \text{uniformly on} \quad [0; s^* + \varepsilon];$$

and for all sufficiently large ν there exists

$$s^*_\nu = \inf \left\{ s > 0 : \begin{array}{l} G(x_p(\tau + \nu^{-1/2}s); \tau + \nu^{-1/2}s) = 0; \\ G'_t|_{(x_p(\tau+\nu^{-1/2}s);\tau+\nu^{-1/2}s)} + \\ + G'_x|_{(x_p(\tau+\nu^{-1/2}s);\tau+\nu^{-1/2}s)} F^r_p\left[x_p(\tau + \nu^{-1/2}s); x_v(\tau + \nu^{-1/2}s); u(\tau + \nu^{-1/2}s); \tau + \nu^{-1/2}s\right) < 0; \end{array} \right\} \qquad (12)$$

such that

$$s^*_\nu \to s^*:$$

Proof: The continuity of the ordinary differential equation solution with respect to the parameters (cf. [19] p. 71 or [50] Th.7 §1) implies that $(y^\nu_p(s); y^\nu_v(s))$ converges to $\to (\bar y_p(s); \bar y_v(s))$ uniformly on $[0; s^* + \varepsilon]$: This follows from Lipschitz condition (8) and assumption1 of Theorem 1.

To prove the second part of Theorem let us define

$$f^0(s) = G'_x(x_p(\tau); \tau)(\bar y_p(s) - x_p(\tau)) + G'_t(x_p(\tau); \tau)s;$$

and

$$f^\nu(s) = G'_x(x_p(\tau + \nu^{-1/2}s); \tau + \nu^{-1/2}s)(y^\nu_p(s) - x_p(\tau)) + G'_t(x_p(\tau + \nu^{-1/2}s); \tau + \nu^{-1/2}s)s:$$

Then

$$\dot f^0(s) = G'_x(x_p(\tau); \tau)\dot{\bar y}_p(s) + G'_t(x_p(\tau); \tau);$$

and

$$\dot f^\nu(s) = G'_x(x_p(\tau + \nu^{-1/2}s); \tau + \nu^{-1/2}s)\dot y^\nu_p(s) + G'_t(x_p(\tau + \nu^{-1/2}s); \tau + \nu^{-1/2}s) + O(\nu^{-1/2}):$$

The proof follows from uniform convergence of $(y^\nu_p(s); \dot y^\nu_p(s))$ to $(\bar y_p(s); \dot{\bar y}_p(s))$ and the lemma below, the proof of which is given in Appendix.

Lemma 1: For any continuous and smooth function $f(\cdot)$ define an exit time

$$\tau^\delta(f) = \begin{cases} \inf\{s > 0 : f(s) \le \delta\}; \\ \infty; \quad \text{if the set is empty:} \end{cases}$$

Let for some function $f^0(\cdot)$ such that $f^0(0) > 0$ the exit time $\tau^0(f^0) < \infty$ and f^0 have the negative derivative at $t = \tau^0(f^0)$: Then $\tau^0(f)$ is continuous at $f^0(\cdot)$ with respect to the topology of uniform convergence. In other words, if $f^n(s) \to f^0(s)$ uniformly on some interval $[0; \tau^0(f^0) + \varepsilon]$; $\varepsilon > 0$; then

$$\tau^0(f^n) \to \tau^0(f^0):$$

By lemma there exists $\tau^0(f^\nu) \to \tau^0(f^0) = s^*$: Then

$$G(x_p(\tau + \nu^{-1/2}s); \tau + \nu^{-1/2}s) - G(x_p(\tau); \tau) =$$

$$= \nu^{-1/2}\left[G'_x(x_p(\tau + \nu^{-1/2}s); \tau + \nu^{-1/2}s)(y^\nu_p(s) - x_p) + G'_t(x_p(\tau + \nu^{-1/2}s); \tau + \nu^{-1/2}s)s\right] + O(\nu^{-1});$$

where $\frac{O(\mu^{-1})}{\mu^{-1/2}} \to 0$, if $\mu \to \infty$ uniformly in $s \in [0, s^* + \varepsilon]$. Since $G(x_p(\tau), \tau) = 0$, we have

$$\mu^{1/2} G(x_p(\tau + \mu^{-1/2}s), \tau + \mu^{-1/2}s) = f^{\mu}(s) + \frac{O(\mu^{-1})}{\mu^{-1/2}}. \tag{13}$$

Due to the negativity of the derivative of f^0 at $\tau^0(f^0)$ there exists δ_0 such that for any $\delta \in (0, \delta_0)$

$$\tau^{\delta}(f^0) < \tau^{-\delta}(f^0) \le s^* + \varepsilon.$$

Then, the uniform convergence of f^{μ} to f^0 for chosen δ and sufficiently large $\mu > \mu_1(\delta)$ yields the inequalities

$$f^{\mu}(\tau^{\delta}(f^0)) \ge \frac{\delta}{2}; \quad f^{\mu}(\tau^{-\delta}(f^0)) \le -\frac{\delta}{2}.$$

At the same time one can choose $\mu_2(\delta)$ such that for $\mu > \mu_2(\delta)$ one obtains $\left|\frac{O(\mu^{-1})}{\mu^{-1/2}}\right| \le \frac{\delta}{4}$. Therefore, from (13) for sufficiently large μ one has

$$\mu^{1/2} G(x_p(\tau + \mu^{-1/2}\tau^{\delta}(f^0)), \tau + \mu^{-1/2}\tau^{\delta}(f^0)) > \frac{\delta}{4}; \quad \mu^{1/2} G(x_p(\tau + \mu^{-1/2}\tau^{-\delta}(f^0)), \tau + \mu^{-1/2}\tau^{-\delta}(f^0)) < -\frac{\delta}{4}.$$

This implies that $G(x_p(\tau + \mu^{-1/2}s), \tau + \mu^{-1/2}s)$ changes the sign within the interval $(\tau^{\delta}(f^0), \tau^{-\delta}(f^0))$. Due to the uniform convergence and the representation (13) there are no other sign inversions at $s < \tau^{\delta}(f^0)$. Therefore, there exists

$$s^*(\mu) \in (\tau^{\delta}(f^0), \tau^{-\delta}(f^0)) \tag{14}$$

satisfying the first equality in (12). The inequality for derivative is evident and convergence of $s^*(\mu)$ to s^* follows from (14) and the arbitrariness of δ. ■

Remark 2: This theorem shows that the limit system (10) demonstrates almost the same behavior as the original one for sufficiently large μ. The main result is the following Corollary which establishes the single jump representation with the aid of the limit system solution.

Corollary 1: For sufficiently small $\varepsilon > 0$ on the interval $[0, \tau + \varepsilon)$, solution of the original system (2), (3) converges to some discontinuous functions $(\bar{x}_p(t), \bar{x}_v(t))$, such that

$$\bar{x}_p(t) = x_p(t); \quad \bar{x}_v(t) = x_v(t); \quad t < \tau;$$

and

$$\bar{x}_p(\tau+) = \lim_{\mu \uparrow \infty} x_p(\tau + \mu^{-1/2}s^*_{\mu}) = x_p(\tau); \quad \bar{x}_v(\tau+) = \lim_{\mu \uparrow \infty} y_v(\tau + \mu^{-1/2}s^*_{\mu}) = \bar{y}_v(s^*).$$

Remark 3: Assumption 1 of Theorem 1 can be relaxed as follows, (cf. [50], Theorem 7 §1)

$$\lim_{\mu \uparrow \infty} \int_0^s \mu^{1/2} F_v^s\left(\frac{y_p - x_p}{\mu^{1/2}} + x_p, y_v, w_1(u), \tau + \mu^{-1/2}u, \mu\right) du = \int_0^s \bar{F}_v^s(y_p, y_v, u, w_1(u), x_p, \tau)\, du; \tag{15}$$

for any y_p, y_v uniformly on $[0, s^* + \varepsilon)$. Condition (15) is much weaker than the Lipschitz one, but it is also sufficient for the uniform convergence.

Remark 4: If in the assumption 2 of Theorem 1 $\varepsilon = 0$, one can establish only the existence of sequence $s^*(\mu) \to s^*$ such that

$$\lim_{\mu \to \infty} G(x_p(\tau + \mu^{-1/2}s^*(\mu)), \tau + \mu^{-1/2}s^*(\mu)) = 0.$$

However, the Corollary 1 is still valid.

B. Representation of the Single-Jump Generalized Solution

Let the system start from the initial condition $x_p(0); x_v(0)$ such that $G(x_p(0); 0) < 0$ and τ be the first point where the system engages the constraint, so that conditions (4) and (5) hold. In order to describe the discontinuity at a point τ, introduce the differential equation

$$\dot{y}_p(s) = \bar{F}_p(y_p(s); y_v(s); u(\tau); x_p(\tau); \tau); \quad \dot{y}_v(s) = \bar{F}_v(y_p(s); y_v(s); s; w_\tau(s); x_p(\tau); \tau); \quad s \in [0; \infty) \tag{16}$$

with initial condition $y_p(0) = x_p(\tau); \quad y_v(0) = x_v(\tau)$ and control signal

$$w_\tau(s) \in W \subset R^k. \tag{17}$$

Let

$$\Psi(\cdot; w_\tau(\cdot); \tau) = \begin{pmatrix} \Psi_p(\cdot; w_\tau(\cdot); \tau) \\ \Psi_v(\cdot; w_\tau(\cdot); \tau) \end{pmatrix} \tag{18}$$

denote a shift operator along the paths of (16). Then the discontinuity of the path $x_v(\cdot)$ at instant $t = \tau$ can be viewed as the result of the action of this shift operator and described by

$$x_v(\tau) = x_v(\tau-) + \Psi_v(x_p(\tau); x_v(\tau-); w_\tau(\cdot); \tau): \tag{19}$$

Relation (19) describes the jump at $t = \tau$ in terms of the equation (16) (the equation of "fast dynamics"), so that if

$$\Phi(\cdot; s; w(\cdot); \tau) = \begin{pmatrix} \Phi_p(\cdot; s; w_\tau(\cdot); \tau) \\ \Phi_v(\cdot; s; w_\tau(\cdot); \tau) \end{pmatrix} \tag{20}$$

is the general solution of (16) with initial condition $y_p(0) = x_p(\tau), \quad y_v(0) = x_v(\tau-),$ then

$$x_v(\tau) = \Phi_v(x_p(\tau); x_v(\tau-); s^*(x_p(\tau); x_v(\tau-)); w_\tau(\cdot); \tau); \tag{21}$$

where

$$s^*(x_p(\tau); x_v(\tau-)) =$$

$$= \inf \left\{ s > 0 : \begin{array}{l} G'_t\big|_{(x_p(\tau);\tau)} s + G'_x\big|_{(x_p(\tau);\tau)} (\Phi_p(x_p(\tau); x_v(\tau-); s; w_\tau(\cdot); \tau) - x_p(\tau)) = 0; \\ G'_t\big|_{(x_p(\tau);\tau)} + G'_x\big|_{(x_p(\tau);\tau)} \times \\ \times \bar{F}_p(\Phi_p(x_p(\tau); x_v(\tau-); s; w_\tau(\cdot); \tau); \Phi_v(x_p(\tau); x_v(\tau-); s; w_\tau(\cdot); \tau); u(\tau); x_p(\tau); \tau) < 0 \end{array} \right\} \tag{22}$$

with $G'_t\big|_{(x_p(\tau);\tau)} s + G'_x\big|_{(x_p(\tau);\tau)} (\Phi_p(x_p(\tau); x_v(\tau-); s; w_\tau(\cdot); \tau) - x_p(\tau)) > 0$ on the interval $(0; s^*(x_p(\tau); x_v(\tau-)))$ and

$$\Psi_v(x_p(\tau); x_v(\tau-); w_\tau(\cdot); \tau) := \Phi_v(x_p(\tau); x_v(\tau-); s^*(x_p(\tau); x_v(\tau-)); w_\tau(\cdot); \tau) - x_v(\tau-): \tag{23}$$

Remark 5: The shift-operator jump representation (23) is admissible for any unconstrained discrete-continuous system that has trajectories robust with respect to variation of impulsive input [36], [38], [58]. For systems with constraints, however, one can not guarantee at the outset that the shift-operator jump representation exists for any initial conditions $x_p(\tau); x_v(\tau-)$ and the impulsive control; $w_\tau(\cdot)$, since the existence of $s^*(x_p(\tau); x_v(\tau-)) < \infty$ can not guaranteed. As follows from the Assumption 2 of Theorem 1, one needs the constraints to be "repulsive". In the next subsection, a useful sufficient condition for the existence of the shift-operator jump representation is given; this condition, however, will be assumed to hold from this point on.

Define the generalized solution of system (2), (3) as a pointwise limit of ordinary solution if $\mu \to \infty$. Then, by using the Corollary 1 and the shift-operator jump representation introduced above, the following is easily derived:

Theorem 2: The generalized solution $\{x_p(t); x_v(t)\}$ satisfies on the interval $[0; \tau + \varepsilon)$ the system of nonlinear generalized differential equations

$$\begin{aligned} \dot{x}_p(t) &= F^r_p(x_p(t); x_v(t); u(t); t); \\ \dot{x}_v(t) &= F^r_v(x_p(t); x_v(t); u(t); t) + \Psi_v(x_p(\tau); x_v(\tau-); w_\tau(\cdot)); \tau)\delta(t - \tau). \end{aligned} \tag{24}$$

C. The Sufficient Condition for the Constraints to be Repulsive

The assumption 2 of Theorem 1 means that "force" $\bar F^1$ has the property to repulse the system from the inhibited domain. Let us express this property in terms of the so-called restitution force. Consider the motion in the area $\{(\bar y_p; s) : G_x^0(x_p(\tau); \tau)[\bar y_p - x_p(\tau)] + G_t^0(x_p(\tau); \tau)s > 0\}$ along the paths of the limit system (10). The term

$$Z(\bar y_p(s); s) = G_x^0(x_p(\tau); \tau)[\bar y_p(s) - x_p(\tau)] + G_t^0(x_p(\tau); \tau)s$$

characterizes the principal part of constraint violation, whereas the restitution force is typically expressed in terms of Z; as $F(s) = \frac{d^2}{ds^2} Z(s)$. Suppose that one can ascertain that this restitution force is of the visco-elastic type and such that

$$\frac{d^2}{ds^2} Z(s) \le -k_1 Z(s) - k_2 \dot Z(s). \tag{25}$$

One can then obtain the following criterion, the proof of which is given in the Appendix.

Proposition 1: Let the restitution force satisfy (25) in the area $Z > 0$. Let also $k_1, k_2 \ge 0$ and satisfy the inequality

$$\left(\frac{k_2}{2}\right)^2 < k_1. \tag{26}$$

Suppose that $Z(0) = 0$ and $\dot Z(0) > 0$. Then there exists

$$s^* = \inf\{s > 0 : Z(s^*) = 0, \dot Z(s^*) < 0\} < \infty.$$

Remark 6: This proposition means that the restitution force (25) guarantees the repulsion in the finite time. Thus, if the limit system (10) satisfies the assumption 2 of Theorem, then the constraint is repulsive.

V. Modeling of Systems with Multiple Controlled Singularities

A. System Model with Multiple Jumps

Having obtained the single jump representation given by Theorems 1 and 2, consider a discrete-continuous dynamical system with behavior representable on the interval $[0, T]$ by a pair of the piecewise continuous functions $(x_p(t), x_v(t)) \in R^n \times R^n$, that satisfies the differential equation

$$\dot x_p(t) = F_p^r(x_p(t), x_v(t), u(t), t); \quad \dot x_v(t) = F_v^r(x_p(t), x_v(t), u(t), t), \tag{27}$$

with a given initial condition $(x_p(0), x_v(0)) \in R^n \times R^n$. In (27) $(x_p(t), x_v(t))$ are the generalized state and velocity

$$u(t) \in U \subset R^m, \tag{28}$$

where U is some compact set and $u(t)$ is a control signal. This equation describes the continuous system evolution until the attainment of the switching surface defined by the relation

$$G(x_p(t), t) = 0. \tag{29}$$

In (27) and (29) functions $F_p^r(x_p, x_v, u, t), F_v^r(x_p, x_v, u, t)$ and $G(x_p, t)$ are continuous in all variables, and $F_{p,v}^r$ are Lipschitzian with respect to (x_p, x_v), so that

$$\|F_{p,v}^r(x_p^1, x_v^1, u, t) - F_{p,v}^r(x_p^2, x_v^2, u, t)\| \le L(\|x_p^1 - x_p^2\| + \|x_v^1 - x_v^2\|) \tag{30}$$

for any $(x_p^1, x_v^1) \in R^n \times R^n$; $(x_p^2, x_v^2) \in R^n \times R^n$; $u \in U$; $t \in [0, T]$ with some positive constant $L < \infty$. Therefore, equation (27) has the unique solution for any Lebesgue measurable control $u(\cdot)$ defined on the whole interval $[0, T]$ with an arbitrary initial condition x_0.

Suppose that originally $G(x_p(0), 0) < 0$, and define a sequence of the intersection times

$$0 < \tau_1 < \ldots < \tau_l < \ldots < T$$

as follows:

$$\tau_1 = \begin{cases} \inf\limits_{0<t\le T} \{t : G(x_p(t), t) > 0\}, \\ \infty, \quad \text{if the set is empty}, \end{cases} \tag{31}$$

and

$$\tau_i = \begin{cases} \inf\limits_{\tau_{i-1}<t\le T} \{t : G(x_p(t),t) > 0\}; \\ \infty, \quad \text{if the set is empty.} \end{cases} \tag{32}$$

As it is seen, $\tau_i, i = 1,2,\ldots$ are the exit times from the domain $\{(x_p,t) : G(x_p,t) \le 0\}$, and since the path $x_p(\cdot)$ is absolutely continuous on intervals (τ_{i-1},τ_i), we have $G(x_p(\tau_i -),\tau_i) = 0$. The times $\tau_i, i = 1,2,\ldots$ are considered to be the times of the path singularity, i.e. the path $x_v(\cdot)$ is considered to be discontinuous at the instant $t = \tau_i$.

B. Representation of the Generalized Solution with Multiple Jumps

Using the shift-operator single jump representation introduced earlier to describe discontinuity at the instant $t = \tau_i$ yields

$$x_v(\tau_i) = x_v(\tau_i -) + \Psi_v(x_p(\tau_i), x_v(\tau_i -), w_{\tau_i}(\cdot), \tau_i) \tag{33}$$

with

$$\Psi_v(x_p(\tau_i), x_v(\tau_i -), w_{\tau_i}(\cdot), \tau_i) := \Phi_v(x_p(\tau_i), x_v(\tau_i -), s^*(x_p(\tau_i), x_v(\tau_i -)), w_{\tau_i}(\cdot), \tau_i) - x_v(\tau_i -) \tag{34}$$

where functions Ψ_v, Φ_v are defined by relations (23), (21), (22). Then, the single-jump description of the discrete-continuous system can be generalized to the case of multiple jumps shown in Figure 5 as follows.

Theorem 3: Suppose that on the interval $[0,T]$ a control signal $u(t) \in U$ is given, the set of instances

$$0 < \tau_1 < \ldots < \tau_i < \ldots < T, \ i \le N < \infty$$

is defined that satisfies (32), and the appropriate shift-operators $\Psi_v(x_p(\tau_i), x_v(\tau_i -), w_{\tau_i}(\cdot), \tau_i)$ augmented by the impulsive control functions $w_{\tau_i}(\cdot)$ are well-defined for each τ_i. Then, if $\mu \to \infty$, the corresponding sequence of ordinary solutions of the system (2), (3) converges everywhere on $[0,T]$ (except, possibly, at the points $\{\tau_i\}$) to the generalized solution $\{x_p(t), x_v(t)\}$ that satisfies on the interval $[0,T]$ the system of nonlinear generalized differential equations

$$\begin{aligned} \dot{x}_p(t) &= F_p^r(x_p(t), x_v(t), u(t), t); \\ \dot{x}_v(t) &= F_v^r(x_p(t), x_v(t), u(t), t) + \sum_{\tau_i \le t} \Psi_v(x_p(\tau_i), x_v(\tau_i -), w_{\tau_i}(\cdot), \tau_i)\delta(t - \tau_i), \end{aligned} \tag{35}$$

or the corresponding equivalent integral form

$$\begin{aligned} x_p(t) &= x_p(0) + \int_0^t F_p^r(x_p(s), x_v(s), u(s), s)ds; \\ x_v(t) &= x_v(0) + \int_0^t F_v^r(x_p(s), x_v(s), u(s), s)ds + \sum_{\tau_i \le t} \Psi_v(x_p(\tau_i), x_v(\tau_i -), w_{\tau_i}(\cdot), \tau_i). \end{aligned} \tag{36}$$

Proof: The proof of this theorem directly follows from Theorem 2 by sequentially taking the limit on each subinterval $[\tau^{\mu}_{i-1},\tau^{\mu}_i) \to [\tau_{i-1},\tau_i)$. The existence of the appropriate $s^*_{\mu}((x^{\mu}_p(\tau^{\mu}_i), x^{\mu}_v(\tau^{\mu}_i -)) \to s^*((x_p(\tau_i), x_v(\tau_i -))$ is ascertained through transversality of all the paths hitting the constraints to the corresponding constraint surfaces and the standard arguments of the continuous dependence of the solution on the initial conditions. The shift-operator representation of jumps also implies the Lipschitzian character of function Ψ_v, thereby guaranteeing the existence and uniqueness of the solution of (35). The details of the proof are standard and are omitted. ■

Remark 7: In this description, the state of the system changes continuously on half-intervals $[0,\tau_1)$, ... $[\tau_{i-1},\tau_i)$; ... $[\tau_N,T]$, and undergoes a sudden change at every instant τ_i. Due to equation (33), the values of these changes depend on the state immediately preceding the jump and the impulsive control signal $w_{\tau_i}(\cdot)$ applied during the singularity phase corresponding to the instant τ_i.

In the next section it is shown how this machinery can be applied to the analysis of the typical mechanical systems with constraints.

Remark 8: Theorem 3 demonstrates that the solutions of the original system (2), (3) and the system of nonlinear generalized differential equations (35) (limit representation) can be made arbitrarily close to each other in a weak-* topology of the space of functions of bounded variation by the appropriate choice of the value of the elasticity coefficient μ.

C. Realization of the impulsive control

The significance of Theorem 3 is in admitting the realization of the desired generalized solutions and the corresponding impulsive control signals obtained on the basis of the limit system in the corresponding original system. Suppose that the generalized solution $\{x_p(t), x_v(t)\}$, the corresponding control signal $u(t) \in U$, the finite set of instants $\{\tau_i\} \in [0, T)$, and the set of the corresponding control functions $\{w_{\tau_i}(s) \in W\}$ defined for each τ_i, have all been obtained on the basis of the limit system representation. For a given finite $\mu < \infty$, by solving system (2) let us define $\tau_1^\mu = \tau_1$ and set $\{x_p^\mu(\tau_1^\mu), x_v^\mu(\tau_1^\mu)\} = \{x_p(\tau_1), x_v(\tau_1)\}$ and then solve system (3) for $t \geq \tau_1$ with controls

$$u(t), \quad w(t, \mu) = \begin{cases} w_{\tau_1}\left(\dfrac{t - \tau_1}{\mu^{-1/2}}\right), & \text{if } t \geq \tau_1, \\ 0, & \text{otherwise.} \end{cases} \tag{37}$$

until the instant $\tau_1^*(\mu)$ satisfying the condition

$$\tau_1^*(\mu) = \inf \left\{ t > \tau_1 : \begin{array}{l} G(x_p^\mu(t), t) = 0, \\ G'_t|_{(x_p^\mu(t), t)} + G'_x|_{(x_p^\mu(t), t)} F_p^r \left(x_p^\mu(t), x_v^\mu(t), u(t), t \right) < 0. \end{array} \right\}. \tag{38}$$

According to Theorem 1 $\tau_1^*(\mu) \to \tau_1$ and $\{x_p^\mu(\tau_1^*(\mu)), x_v^\mu(\tau_1^*(\mu))\} \to \{x_p(\tau_1), x_v(\tau_1)\}$ as $\mu \to \infty$. Then, continue the solution as that of system 2 for $t \geq \tau_1^*(\mu)$. As follows from the continuity of the solutions of differential equations, there exists τ_2^μ satisfying the condition

$$\tau_2^\mu = \begin{cases} \inf\limits_{\tau_1^*(\mu) < t \leq T} \{t : G(x_p^\mu(t), t) > 0\}, \\ \infty, \quad \text{if the set is empty.} \end{cases} \tag{39}$$

such that $\tau_2^\mu \to \tau_2$ and $\{x_p^\mu(\tau_2^\mu), x_v^\mu(\tau_2^\mu)\} \to \{x_p(\tau_2), x_v(\tau_2 -)\}$ as $\mu \to \infty$. Then, repeat the procedure described above with the control signals defined for $t \geq \tau_2^\mu$ as follows:

$$u(t), \quad w(t, \mu) = \begin{cases} w_{\tau_2}\left(\dfrac{t - \tau_2^\mu}{\mu^{-1/2}}\right), & \text{if } t \geq \tau_2^\mu, \\ 0, & \text{otherwise;} \end{cases} \tag{40}$$

and so on. Then, according to Theorem 3, the resulting solution $\{x_p^\mu(t), x_v^\mu(t)\} \to \{x_p(t), x_v(t)\}$, i.e., the solution of the original system converges to that of the limit system, uniformly in μ as $\mu \to \infty$ at all points of continuity.

VI. Multi-Scale and Limit Representation of Mechanical Systems with Controlled Constraints: an analysis of motivating examples

Let us apply the description of motion given by the theorems formulated in the previous section to two mechanical systems introduced in Section II.

A. Example 1: Collision Mapping in the Motion of a Ball Colliding with a Moving Racket

In Example 1 given Section II, the system has the state vector $z = (x_p^1, x_p^2, X_p, x_v^1, x_v^2, X_v)$, where (x_p^1, x_p^2) are the horizontal and the vertical coordinates of the moving ball, (x_v^1, x_v^2) are the corresponding velocities, and (X_p, X_v) are the coordinate and velocity of the obstacle surface in the vertical direction. Then, the constraint which defines the free motion area is given by the relation

$$G(z) = X_p - x_p^2 \leq 0. \tag{41}$$

1. General motion equations. First, consider the motion along the vertical axis only. Suppose that in the area free of constraint the motion is described by the equations

$$\begin{aligned} \dot{x}_p^1(t) &= x_v^1(t), \quad \dot{x}_v^1(t) = 0, \\ \dot{x}_p^2(t) &= x_v^2(t), \quad \dot{x}_v^2(t) = -g, \\ \dot{X}_p(t) &= X_v(t), \quad \dot{X}_v(t) = \frac{F(t)}{M}. \end{aligned} \tag{42}$$

In the inhibited area the motion is described by the equation

$$\begin{aligned}
&\dot{x}^1_p(t) = x^1_v(t); \quad \dot{x}^1_v(t) = -\mu\kappa\frac{F^s(z(t);\kappa)}{m}\,\mathrm{sign}\,x^1_v(s);\\
&\dot{x}^2_p(t) = x^2_v(t); \quad \dot{x}^2_v(t) = -g - \kappa\frac{F^s(z(t);\kappa)}{m};\\
&\dot{X}_p(t) = X_v(t); \quad \dot{X}_v(t) = \frac{F(t)}{M} + \kappa\frac{F^s(z(t);\kappa)}{M}:
\end{aligned} \tag{43}$$

where

- $F^s(z(t);\kappa)$ is a visco-elastic force during the contact of ball and racket described by the relation

$$F^s(z(t);\kappa) = -(x^2_p(t) - X_p(t)) - 2\kappa^{-1/2}\xi(x^2_v(t) - X_v(t)); \tag{44}$$

- $m; M$ are the masses of ball and racket, respectively,
- g is the gravitational acceleration,
- $\mu \geq 0$ is the dry friction coefficient,
- $F(t)$ such that $|F(t)| \leq F_0 < \infty$, is an external control force acting on the racket,
- $\kappa > 0$ is the elasticity coefficient and $0 \leq \xi \leq 1$ is the damping.

Equations (42),(43) describe the continuous motion in case of $\kappa < \infty$. The objective, however, is to obtain the velocity jump representation corresponding the limit motion as $\kappa \to \infty$:

2. Jump representation in the vertical direction. Since the motion in the vertical direction does not depend on coordinates $(x^1_p; x^1_v)$ one can obtain the jump representation independently. First, consider the motion in the vertical direction with the corresponding reduced state vector $\tilde{z}(t) = (x^2_p; X_p; x^2_v; X_v)$. Applying Theorem 1 and calculating $\tilde{F}$ by formula (9) yields the following system (cf. eq. (10)) for new variables $(\tilde{y}^2_p(s); \tilde{Y}_p(s); \tilde{y}^2_v(s); \tilde{Y}_v(s))$ describing the motion in the enlarged space-time scale:

$$\frac{d}{ds}\begin{pmatrix} \tilde{y}^2_p(s) \\ \tilde{Y}_p(s) \\ \tilde{y}^2_v(s) \\ \tilde{Y}_v(s) \end{pmatrix} = \begin{pmatrix} \tilde{y}^2_v(s) \\ \tilde{Y}_v(s) \\ -\frac{1}{m}\left(\tilde{y}^2_p(s) - \tilde{Y}_p(s)\right) - 2\xi\left(\tilde{y}^2_v(s) - \tilde{Y}_v(s)\right) \\ \frac{1}{M}\left(\tilde{y}^2_p(s) - \tilde{Y}_p(s)\right) - 2\xi\left(\tilde{y}^2_v(s) - \tilde{Y}_v(s)\right) \end{pmatrix}: \tag{45}$$

The system (45) has to be solved for $s > 0$ with initial conditions

$$\tilde{y}^2_p(0) = \tilde{Y}_p(0) = x^2_p(\tau); \quad \tilde{y}^2_v(0) = x^2_v(\tau) > \tilde{Y}_v(0) = X_v(\tau); \tag{46}$$

until the exit time defined by the relation (11)

$$s^* = \inf\left\{ s > 0 : \begin{array}{l} \tilde{y}^2_p(s) - \tilde{Y}_p(s) = 0; \\ \tilde{y}^2_v(s) - \tilde{Y}_v(s) < 0 \end{array} \right\}: \tag{47}$$

The variable

$$Z(s) = \tilde{y}^2_p(s) - \tilde{Y}_p(s);$$

which characterizes the constraint violation satisfies the equation

$$\frac{d^2}{ds^2}Z(s) = \frac{M+m}{Mm}\left(-Z(s) - 2\xi\frac{d}{ds}Z(s)\right);$$

with initial conditions

$$Z(0) = 0; \quad \frac{d}{ds}Z(0) > 0:$$

Therefore, according to Proposition 1 this constraint is repulsive if the condition

$$|\xi| < \left(\frac{mM}{M+m}\right)^{1/2} = \frac{1}{k} \tag{48}$$

holds. In this case there exists

$$s^{*} = \frac{\pi}{\sqrt{k - k^2\xi^2}}$$

satisfying (47) and by Corollary 1 one can calculate the jumps of variables $x_v^2(\tau)$ and $X_v(\tau)$: They are equal to

$$\begin{aligned} \Delta x_v^2(\tau) &= -\frac{M(1+k_r)}{M+m}\left[x_v^2(\tau-) - X_v(\tau-)\right]; \\ \Delta X_v(\tau) &= \frac{m(1+k_r)}{M+m}\left[x_v^2(\tau-) - X_v(\tau-)\right]; \end{aligned} \tag{49}$$

where the restitution coefficient k_r is equal to

$$k_r = \exp\left\{-\frac{\pi k\xi}{\sqrt{k - k^2\xi^2}}\right\}.$$

3. Motion in the horizontal direction with dry friction. By using the space-time transformation of $(x_p^1; x_v^1)$ we obtain the following system of equations in new space-time scale

$$\begin{pmatrix} \dot{\tilde{y}}_p^1(s) \\ \dot{\tilde{y}}_v^1(s) \end{pmatrix} = \begin{pmatrix} \tilde{y}_v^1(s) \\ -\gamma \dot{\tilde{y}}_v^2(s)\,\mathrm{sign}\{\tilde{y}_v^1(s)\}; \end{pmatrix} \tag{50}$$

where by definition $\mathrm{sign}\{0\} = 0$. Solution of (50) has the form

$$\tilde{y}_v^1(s) = \tilde{y}_v^1(0) - \gamma \int_0^{\min\{s;\tau^*\}} \dot{\tilde{y}}_v^2(\tau)d\tau = \tilde{y}_v^1(0) - \gamma[\tilde{y}_v^2(\min\{s;\tau^*\}) - \tilde{y}_v^2(0)];$$

where

$$\tau^{*} = \inf\{\tau > 0 : \tilde{y}_v^1(\tau) = 0\}.$$

Therefore,

$$\tilde{y}_v^1(s^*) = \tilde{y}_v^1(0) - \gamma[\tilde{y}_v^2(\min\{s^*;\tau^*\}) - \tilde{y}_v^2(0)];$$

and taking into account the initial condition $\tilde{y}_v^1(0) = x_v^1(\tau-)$ by Corollary 1 one can calculate velocity increment in the horizontal direction as

$$\Delta x_v^1(\tau) = \begin{cases} -\gamma\Delta x_v^2(\tau); & \text{if } [x_v^1(\tau-) - \gamma\Delta x_v^2(\tau)]x_v^1(\tau-) \ge 0; \\ -x_v^1(\tau-); & \text{otherwise}; \end{cases} \tag{51}$$

where $\Delta x_v^2(\tau)$ is defined by the relation (49).

The next example shows another type of introduction of the impulsive action into the singular phase.

B. Example 2: Impact with the rotating surface

Let the dynamics of the ball be described by a pair of vectors $(x_p; x_v)$; where $x_p \in R^3$ correspond to the position and $x_v \in R^3$ to the velocity, so that the nonsingular phase is described by the equation

$$\dot{x}_p(t) = x_v(t); \quad \dot{x}_v(t) = 0.$$

The position of the racket surface plane is given by the relation $\langle y; n(t;\mu)\rangle = 0$; where $\langle \cdot;\cdot \rangle$ is the scalar product, μ is the elasticity coefficient, and $n(t)$ is a unit vector of normal to the surface. For simplicity, consider the case when the center of the rotation coincides with the point of contact and the rotation axis is orthogonal to the plane formed by vectors x_v and n. In the singular phase, which takes place whenever the ball hits the surface, the contact force is proportional to the value of the surface deformation and directed perpendicular to the surface, so that if

$$G(x_p(t); t) = \langle x_p(t); n(t;\mu)\rangle < 0;$$

the equations have the form

$$\dot x_p(t) = x_v(t);\quad \dot x_v(t) = -\varepsilon n(t;\varepsilon) < x_p(t); n(t;\varepsilon) > :$$

One can complete this system with the differential equation for $n(t;\varepsilon)$

$$\dot n(t;\varepsilon) = n(t;\varepsilon) \times \omega(t;\varepsilon)$$

where $a \times b$ denotes the vector product, and $\omega(t;\varepsilon)$ is the angular velocity of the surface rotation. This velocity could be interpreted as an impulsive control that abruptly changes the angle of the surface during the contact phase. Therefore, this velocity admits the representation

$$\omega(t;\varepsilon) = \omega_0 \varepsilon^{1/2} w\left(\frac{t-\tau}{\varepsilon^{-1/2}}\right)$$

where ω_0 is the unit vector directed along the rotation axis, $w(\cdot)$ is an impulsive control, and τ is the impact time. The use of the transformation (6)

$$\begin{aligned} y^\varepsilon_p(s) &= x_p(\tau) + \varepsilon^{1/2}[x_p(\tau + \varepsilon^{-1/2}s) - x_p(\tau)]; \\ y^\varepsilon_v(s) &= x_v(\tau + \varepsilon^{-1/2}s);\quad n^\varepsilon(s) = n(\tau + \varepsilon^{-1/2}s;\varepsilon) \\ t &= \tau + \varepsilon^{-1/2}s; \end{aligned} \tag{52}$$

yields the following limit system for limit variables $(\hat y_p(s); \hat y_v(s); \hat n(s))$ defined for $s > 0$:

$$\begin{pmatrix} \dot{\hat y}_p(s) \\ \dot{\hat y}_v(s) \\ \dot{\hat n}(s) \end{pmatrix} = \begin{pmatrix} \hat y_v(s) \\ -\hat n(s) < \hat y_p(s); \hat n(s) > \\ (\hat n(s) \times \omega_0) w(s) \end{pmatrix} : \tag{53}$$

This system has to be solved with the initial conditions

$$\hat y_p(0) = O;\quad \hat y_v(0) = x_v(\tau^-);\quad \hat n(0) = n(\tau^-);\quad \text{such that}\quad < x_v(\tau^-); n(\tau^-) >< 0;$$

until the time

$$s^* = \inf\left\{ s > 0 : \begin{array}{c} < \hat y_p(s); \hat n(s) > = 0 \\ \frac{d}{ds} < \hat y_p(s); \hat n(s) > > 0 \end{array} \right\} : \tag{54}$$

If

$$\hat n(0) = n(\tau^-) = \begin{pmatrix} n_1(\tau^-) \\ n_2(\tau^-) \\ 0 \end{pmatrix}$$

is given, then the solution of the third equation is equal to

$$\hat n(s) = \begin{pmatrix} n_1(\tau^-)\cos\{\phi(s)\} + n_2(\tau^-)\sin\{\phi(s)\} \\ -n_1(\tau^-)\sin\{\phi(s)\} + n_2(\tau^-)\cos\{\phi(s)\} \\ 0 \end{pmatrix}$$

where

$$\phi(s) = \int_0^s w(u)du$$

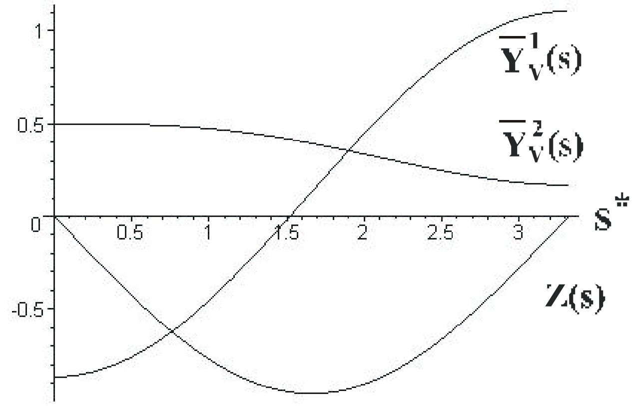

Fig. 6. Solution of the limit system. Clockwise rotation.

is the rotation angle of the contacting surface. Therefore, if the control law $w(\cdot)$ is given, one obtains a linear system of differential equations

$$\begin{pmatrix} \dot{\hat y}_p(s) \\ \dot{\hat y}_v(s) \end{pmatrix} = \begin{pmatrix} \hat y_v(s) \\ -\ddot{\hat n}(s) < \hat y_p(s); \hat n(s) > \end{pmatrix} = \begin{pmatrix} \hat y_v(s) \\ A(\phi(s); \hat n(\tau_-))\hat y_p(s) \end{pmatrix} \qquad (55)$$

for $(\hat y_p; \hat y_v)$; where $A(\phi; n)$ is some matrix-valued function depending on the current value of the rotation angle, $\phi(s)$, and the initial orientation, $n(\tau_-)$, of the plane. The general solution of this system has the form

$$\begin{pmatrix} \hat y_p(s) \\ \hat y_v(s) \end{pmatrix} = \Phi(s; 0; \phi(\cdot)) \begin{pmatrix} \hat y_p(0) \\ \hat y_v(0) \end{pmatrix}:$$

Matrix $\Phi(s; 0; \phi(\cdot))$ can be represented as

$$\Phi(s; 0; \phi(\cdot)) = \begin{pmatrix} \Phi_{pp} & \Phi_{pv} \\ \Phi_{vp} & \Phi_{vv} \end{pmatrix}(s; 0; \phi(\cdot)):$$

If there exists an exit time s^* defined by the relation (54), the collision mapping in this model is then calculated as

$$\Delta x_v(\tau) = \hat y_v(s^*) - \hat y_v(0) = \{\Phi_{vv}(s^*; 0; \phi(\cdot)) - I\} x_v(\tau_-):$$

The numerical examples demonstrate the calculation of collision mapping for the following data:

- $$x_v^1(\tau_-) = -\frac{\sqrt{3}}{2}; \quad x_v^2(\tau_-) = \frac{1}{2};$$
- $$n^1(\tau) = 1; \quad n^2(\tau) = 0;$$
- $$w(s) = \hat w = -0{:}1; \quad \text{and} \quad w(s) = \hat w = 0{:}1:$$

Figures 6 and 7 show the results of the solution calculation of the limit system (55) for positive and negative direction of the surface rotation, respectively.

Using the curves obtained, one can define s^* as the first point where $Z(s) = < \hat y(s); n(s) > = 0$, and then define

$$\Delta\phi(\tau) = \hat w s^*; \quad \Delta x_v^1(\tau) = \hat y_v^1(s^*) - x_v^1(\tau_-); \quad \Delta x_v^2(\tau) = \hat y_v^2(s^*) - x_v^2(\tau_-);$$
$$\Delta V(\tau) = \sqrt{\Delta (x_v^1)^1(\tau) + (\Delta x_v^2)^2(\tau)}:$$

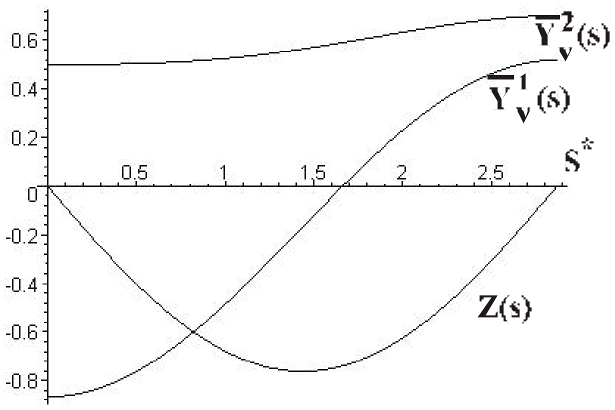

Fig. 7. Solution of the limit system. Anti-clockwise rotation.

In the first case corresponding to the clockwise rotation

$$s^* = 3.32; \quad \Delta\Phi(\tau) = -0.332\,\mathrm{rad}; \quad \Delta x_v^1(\tau) = 1.97; \quad x_v^2(\tau) = -0.3; \; \Delta V(\tau) = 0.1 > 0;$$

and in the second case corresponding to the anti-clockwise rotation

$$s^* = 2.87; \quad \Delta\Phi(\tau) = 0.287\,\mathrm{rad}; \quad \Delta x_v^1(\tau) = 1.37; \quad x_v^2(\tau) = 0.2; \; \Delta V(\tau) = -0.12 < 0.$$

Thus, fully inclusive as well as simplified representations of systems with controlled collisions are obtained.

VII. Conclusions

A new class of systems: dynamical systems with controlled singularities is proposed. A subset of this class - systems with controlled elastic impacts - is considered. For the latter subset, a mathematical framework is developed that provides consistent description of controlled impact dynamics and yields control-oriented models suitable for system redesign, motion planning, and control law synthesis. The mathematical framework includes: a system of differential equations with unbounded impulsive rhs, or original system, that accommodates controlled collisions, a topological map that regularizes the original system by mapping it into the limit representation, and the limit representation that has bounded control signals and coefficients of the delta-functions in its rhs. It is demonstrated that the corresponding solutions of the original and the limit representations can be made arbitrarily close to each other uniformly except, possibly, in the vicinities of the jump points by the appropriate setting of the value of the elasticity coefficient in the original system. A new type of description of the impulsive action in the form of the controlled shift operator along the trajectories of the equation of fast dynamics in the singular phase is introduced and incorporated into the system limit representation; this operator can be viewed as active collision mapping, replacing the traditional one. Implementation of the control laws is shown to be accomplished by simply substituting the time-rescaled bounded control signals found through the use of the limit representation into the actual physical system, resulting in the behavior of the latter close to that of the limit system for sufficiently large values of the elasticity coefficient and sufficiently accurate original system description.

Two detailed examples are given which demonstrate the use of the framework developed for the introduction of control actions into singular phase and rigorous derivation of the limit description of the system. The concepts also offer a much more general and convenient framework for representing various types of mechanical collisions even in the absence of the deliberately introduced impulsive actions.

Appendix

Proof: [Proof of Lemma 1]Due to the negativity of derivative at $\tau^0(f^0)$ there exists $\delta_0 > 0$ such that $\tau^{\delta}(f^0) < \tau^0(f^0) < \tau^{-\delta}(f^0)$ for any $0 < \delta < \delta_0$. Moreover, $\tau^{\delta}(f^0) \uparrow \tau^0(f^0)$; and $\tau^{-\delta}(f^0) \downarrow \tau^0(f^0)$ if $\delta \to 0$. Due to the uniform convergence of $\{f^n(\cdot)\}$ to $f^0(\cdot)$ for any $\delta < \delta_0$ one can choose $n(\delta)$ such that for $n \geq n(\delta)$

$$f^n(s) > f^0(\tau^{\delta}(f^0)) - \frac{\delta}{2} = \frac{\delta}{2}; \quad \text{for} \quad s \leq \tau^{\delta}(f^0)$$

and

$$f^n(s) < f^0(\tau^{-\delta}(f^0)) - \frac{\delta}{2} + \frac{\delta}{2}; \quad \text{for} \quad s = \tau^{-\delta}(f^0)$$

Hence by continuity of f^n for $n \geq n(\delta)$, we have

$$\tau^0(f^n) \in (\tau^{\delta}; \tau^{-\delta})$$

which establishes the convergence due to the arbitrariness of δ. ■

Proof: [Proof of Propositionof 1]Function $Z(s)$ satisfies the equation

$$\frac{d^2}{ds^2}Z(s) = f(s) - k_1 Z(s) - k_2 \dot{Z}(s);$$

with some function $f(s) \leq 0$. This equation has an explicit solution

$$Z(s) = e^{-\lambda s}\left[Z(0)\frac{\sin \omega s}{\omega} + \int_0^s e^{\lambda \tau}\frac{\sin \omega (s - \tau)}{\omega} f(\tau)d\tau\right] \tag{56}$$

where

$$\omega^2 = k_1 - \left(\frac{k_2}{2}\right)^2; \quad \lambda = \frac{k_2}{2}.$$

It is easily seen that $Z\left(\frac{\pi}{\omega}\right) \leq 0$. Consider first the case when the Lebesgue measure

$$\mathrm{mes}\,\{s : f(s) < 0\} > 0.$$

Then

$$\int_0^{\pi/\omega} e^{\lambda \tau}\frac{\sin \omega (s - \tau)}{\omega} f(\tau)d\tau < 0,$$

due to positivity of $\sin \omega (s - \tau)$ and negativity of f. Since $\dot{Z}(0) > 0$, there exist an interval, $[0, \varepsilon)$ where $Z(s) > 0$. Therefore, there exists $s^* \in \left(0, \frac{\pi}{\omega}\right]$ such that $Z(s^*) = 0$ and $Z(s) > 0$ for $s \in [0, s^*)$. At the same time

$$Z(s^*) = e^{-\lambda s^*}\left[Z(0)\cos \omega s^* + \int_0^{s^*} e^{\lambda \tau}\cos \omega (s^* - \tau) f(\tau)d\tau\right]; \tag{57}$$

thus taking into account (56) and (57) we obtain

$$Z(s^*)\sin \omega s^* = e^{-\lambda s^*}\int_0^{s^*} e^{\lambda \tau}\sin \omega \tau f(\tau)d\tau < 0.$$

This inequality implies that $Z(s^*) < 0$ if $\sin \omega s^* \neq 0$. Otherwise, $\sin \omega s^* = 0$ yields $s^* = \frac{\pi}{\omega}$, and $\int_0^{s^*} e^{\lambda \tau}\sin \omega \tau f(\tau)d\tau = 0$. In this case $f(s) = 0$ almost everywhere on $[0, s^*]$ and as follows from (57)

$$Z(s^*) = -e^{-\lambda s^*} Z(0) < 0.$$

■

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