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OPTIMIZATION OF OBSERVATIONS: A STOCHASTIC CONTROL APPROACH*

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Abstract. We study a stochastic control problem for the optimization of observations in a partially observable stochastic system. Using a method of discontinuous time transformation, we associate with the original problem with unbounded controls a problem that has bounded controls. This latter problem allows us to construct nearly optimal nonanticipative Lipschitz Markov controls with finite observation power for the original problem. Since the controlled observation equation may degenerate, we also derive a corresponding filtering result and show a separation property of the optimal controls.

Key words. partially observed stochastic systems, observation control, nonlinear filtering, separation principle, discontinuous time transformation

AMS subject classifications. Primary, 93E03, 93E20; Secondary, 93E11, 60G35, 49N25

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Introduction. The most common way to formulate a control problem is to let the control affect only the evolution of the state while possible partial observations of the state are supposed to be continuously available.

Many practical situations, however, lead to the possibility that the observations also can be controlled in a way that affects both their timing as well as their quality. This then leads to a control problem where one tries to choose the control in a way to maximize the information content of the observations regarding the state while at the same time also taking into account a possible penalization of the control effort. Since the information content of the observations can be measured by the state estimation covariance, maximization of the information content can be obtained by minimizing this estimation covariance.

Problems of optimization of observations were mainly studied in the East (see, e.g., [1], [2], [9], [11]); a first study in the West appears in [4]. In a stochastic context only the linear case has been studied so far. More precisely, in [9] the authors consider the following linear model:

$$dx_t = a_t x_t\,dt + b_t\,dw_t^{(1)}, \tag{0.1a}$$

$$dy_t = A_t(u_t) x_t\,dt + B_t\,dw_t^{(2)}, \tag{0.1b}$$

where u_t is the observation control and $(w_t^{(1)})$ and $(w_t^{(2)})$ are independent standard Wiener processes. In this linear case the estimation covariance γ does not depend on the observations and so the optimal control becomes a deterministic time function.

The purpose of this paper is an attempt to extend the investigations to the nonlinear case. As a first step in this direction we consider a model related to (0.1), where

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the coefficients depend on an unknown parameter and the observation noise is more realistically considered as an endogenous noise induced by the observations themselves; the observation power is restricted to be finite at all times with bounded total observation energy. Contrary to the linear case, the observation covariance here depends on the observations and the control problem itself becomes a stochastic control problem.

The structure of the paper is as follows: in section 1 we present our control model and study the associated filter problem which, due to the possibility that the (controlled) observation equation may degenerate, cannot be approached directly by standard techniques. In section 2 we then formulate the full control problem and show a separation property, namely, that among the optimal controls there is one depending on the observations through the filter values. This control problem is a nonstandard nonlinear problem with finite but unbounded controls. In section 3, using a stochastic version of the so-called method of discontinuous time transformation (see [10] for a deterministic context), we therefore derive an auxiliary problem with bounded controls and study the relationship between the original and the auxiliary control problems. While the auxiliary problem can be shown to admit an optimal solution, for the original problem there may not exist an optimal nonanticipative solution. On the other hand, the auxiliary problem gives also the possibility to derive a nearly optimal nonanticipative Lipschitz Markov (feedback) control for the original problem. Finally, in the concluding remarks we recall some of the delicate points of our approach.

1. The model and the associated filter process.

1.1. The model. On a given finite time interval $[0,T]$ consider a partially observed process (x_t, y_t) that satisfies the following linear system, parametrized by an unknown parameter θ and with a control in the observations:

$$x_t = x_0 + \int_0^t a_s(\theta)x_s\,ds + \int_0^t b_s(\theta)\,dw_s^{(1)}, \tag{1.1a}$$

$$y_t = \int_0^t A_s(\theta)x_s\,dv_s + \eta_t. \tag{1.1b}$$

In this system, where for simplicity of presentation we consider all processes to be scalar valued, $(w_t^{(1)})$ is a standard Wiener process with respect to a given filtration $(\mathcal{F}_t)$ with $\mathcal{F}_t \supseteq \mathcal{F}_t^y := \sigma\{y_s,\, s \leq t\}$; x_0 is $\mathcal{F}_0$-measurable with distribution $\mathcal{N}(m_0, \sigma_0)$ and independent of $(w_t^{(1)})$. The observation control process (v_t) is an $\mathcal{F}_t^y$-adapted absolutely continuous and almost surely (a.s.) nondecreasing process with $v_0 = 0$ that thus has almost everywhere (a.e.) a derivative $u_t = \dot{v}_t$ that we assume satisfies the restrictions

$$0 \leq u_t < +\infty \qquad \text{(finite observation power)}, \tag{1.2a}$$

$$\int_0^T u_t dt = v_T \leq M < +\infty \qquad \text{(finite observation energy)}. \tag{1.2b}$$

The additive observation disturbance consists of an endogenously induced noise, due to the observation itself and represented by the (conditionally) Gaussian $\mathcal{F}_t^y$-martingale (η_t), whose quadratic variation satisfies the compatibility condition

$$\langle \eta \rangle_t = B^2\, v_t \qquad (B \neq 0) \tag{1.3}$$

and is independent of $(w_t^{(1)})$ and x_0. Although other reasonable models could possibly be posited, in this first approach to the control of the observations in the nonlinear stochastic case we assume a compatibility condition in the form of (1.3), which implicitly states that drift and noise in the observation equation are both linear in the control (see the ensuing equivalent representation of model (1.1) in (1.6) below). By considering more complex situations, the control may enter the diffusion term in a nonmultiplicative way so that the absolute continuity of the drift with respect to the quadratic variation of the noise in the observation equation may be lost with ensuing additional problems for the filter. Note also that we may let the constant B in (1.3) be substituted by a time function B_t; dividing the observations by the (known) function B_t then reduces this more general case to the one treated here.

As a consequence of (1.3), there exists an $\mathcal{F}_t$-standard Wiener process $(w_t^{(2)})$, independent of $(w_t^{(1)})$ and x_0, so that the following representation holds:

$$\eta_t = B \int_0^t (u_s)^{1/2} dw_s^{(2)}. \tag{1.4}$$

For this representation (1.4) and analogous ones later, as is usually done we implicitly assume that, where necessary, the underlying probability space is sufficiently enlarged to support all required Wiener processes. (For an explicit construction of such an enlargement see, e.g., section 1.4.4 in [5].)

The dependence of u_t on the observation history implies that (1.1b) is actually an equation in (y_t). To ensure that (1.1b) is well defined, we shall thus assume that u_t as a function of the observation history $y_0^t := \{y_s, s \le t\}$ is such that it satisfies a Lipschitz property in the sense that for a nondecreasing and right continuous function $K(t)$, $0 \le K(t) \le 1$, and some nonnegative constants L_1, L_2 we have for all $t \ge 0$

$$|u_t(y_0^t) - u_t(\tilde{y}_0^t)\,|^2 \le L_1 \int_0^t |y_s - \tilde{y}_s|^2 dK(s) + L_2 |y_t - \tilde{y}_t|^2. \tag{1.5}$$

Furthermore, taking the Bayesian point of view, the unknown parameter θ is considered an $\mathcal{F}_0$-measurable random variable, independent of $x_0, (w_t^{(1)})$, and (η_t) and taking a finite number of possible values θ^i $(i = 1, \dots, k)$ with prior probabilities $p_i = P(\theta = \theta^i)$. Finally, $a_t(\theta), b_t(\theta)$, and $A_t(\theta)$ are continuous and bounded functions of t for all θ.

In the setting just described, system (1.1) can equivalently be represented as

$$dx_t = a_t(\theta) x_t\, dt + b_t(\theta)\, dw_t^{(1)}, \tag{1.6a}$$

$$dy_t = A_t(\theta) x_t u_t\, dt + B \cdot (u_t)^{1/2} dw_t^{(2)}, \quad y_0 = 0, \tag{1.6b}$$

$$d\theta = 0. \tag{1.6c}$$

Remark 1.1. Since the value of u_t corresponds to the power applied to the observation of the signal x_t, conditions (1.2) imply that we require this power to be finite at each t with bounded total observation energy, which indeed corresponds to the actual physical situation. The fact that the additive observation noise in (1.1b), or equivalently in (1.6b), is given by an endogenously generated noise due to the observation itself justifies the assumption of a compatibility condition such as (1.3). The Gaussian assumption for this noise can be justified for those cases when the power u_t, applied for observing x_t, is sufficiently large; take, e.g., an optical noise that is Poisson with intensity proportional to the power of the observation so that for large values of this power it can be approximated by a Gaussian. In a sense, the observation noise in (1.1b) or (1.6b) is thus a minimum-level noise, to which some independent exogenous Gaussian noise could possibly be added as well. Note finally that the observation inaccuracy is due not only to the additive observation noise but also to the averaging of the signal x_t as implied by the first term on the right of the observation equation (1.1b) or (1.6b). The averaging due to the choice of v_t (equivalently of u_t) thus affects both the timing and the quality of the observations: in the limit, when the observation power tends to infinity, u_t tends to a δ-function thus determining only the timing of the observations with no averaging of the signal. Note also that the observation power may tend to infinity, while the total observation energy remains still bounded by M.

Before going on to describe the full control problem, we study the filter process associated with the given partially observed control model.

1.2. The filter process. For $i = 1, \ldots, k$ consider

$$m_t^i := E\{x_t | \, \mathcal{F}_t^y, \theta^i\}, \tag{1.7a}$$

$$\gamma_t^i := E\{(x_t - m_t^i)^2 | \, \mathcal{F}_t^y, \theta^i\}, \tag{1.7b}$$

$$\pi_t^i := P\{\theta = \theta^i | \, \mathcal{F}_t^y\}, \tag{1.7c}$$

and let the "filter process" X_t be given by the following set of triplets:

$$X_t := \{m_t^i, \gamma_t^i, \pi_t^i\}_{i=1,\ldots,k}. \tag{1.8}$$

The main purposes of this subsection are to derive a stochastic differential equation for X_t and to show that, under the assumptions of subsection 1.1, it has a unique solution. We point out that these results will not simply be a direct application of known filtering results since, due to the possibility that u_t may be equal to zero on intervals of positive length, the observation equation may degenerate. The main result of this section is the following theorem.

THEOREM 1.2. *The filter process* (X_t) *in* (1.8) *satisfies, for a given control* v_t *(equivalently* u_t*) and* $i = 1, \ldots, k$,

$$dm_t^i = a_t(\theta^i) m_t^i dt + B^{-2} A_t(\theta^i) \gamma_t^i \, [dy_t - A_t(\theta^i) m_t^i dv_t], \quad m_0^i = m_0 = E(x_0), \tag{1.9a}$$

$$d\gamma_t^i = 2a_t(\theta^i)\gamma_t^i dt + b_t^2(\theta^i) dt - B^{-2}[A_t(\theta^i)\gamma_t^i]^2 dv_t, \quad \gamma_0^i = \sigma_0 = \mathrm{Cov}(x_0), \tag{1.9b}$$

$$d\pi_t^i = \pi_t^i \left[A_t(\theta^i) m_t^i - \sum_{j=1}^k \pi_t^j A_t(\theta^j) m_t^j \right] B^{-2} d\xi_t, \quad \pi_0^i = p_i, \tag{1.9c}$$

where

$$\xi_t := \int_0^t \left[dy_s - \sum_{j=1}^k \pi_s^j A_s(\theta^j) m_s^j dv_s \right] \tag{1.10}$$

is an $\mathcal{F}_t^y$-conditionally Gaussian martingale with quadratic variation

$$\langle \xi \rangle_t = v_t(y_0^t). \tag{1.11}$$

Furthermore, the system (1.9) *can be represented in compact form as*

$$dX_t = F_t(X_t)dt + B_t(X_t)u_t dt + G_t(X_t)u_t^{1/2} dw_t^X \tag{1.12}$$

for suitable functions F, B, and G related to the coefficients in (1.9) *and where (w_t^X) is an $\mathcal{F}_t$-standard Wiener process. Finally, for any given control (v_t) or (u_t), the solution of* (1.9) (*or* (1.12)) *is unique.*

To prove this theorem (the proof will be given below) we shall need an intermediate result for an auxiliary filtering problem that will allow us to cope with the possible degeneracy of the original (controlled) filtering model. To derive the auxiliary problem we use an absolutely continuous time transformation. More precisely, for a given control v_t (recall that $v_0 = 0$) let

$$\Gamma_t := v_t + \int_0^t I\{s : \dot{v}_s = 0\}\, ds, \tag{1.13}$$

which is an $\mathcal{F}_t^y$-adapted absolutely continuous process with strictly positive derivative and satisfying $\Gamma_T \leq M + T$ (see (1.2)). On the interval $[0, \Gamma_T]$ it admits thus the inverse function

$$\nu_s = \inf\{\tau : \Gamma_\tau > s\} = \inf\{\tau : \Gamma_\tau = s\}, \tag{1.14}$$

which satisfies $0 \leq \nu_s \leq T$ and is absolutely continuous with

$$\dot{\nu}_s = \frac{1}{\dot{\Gamma}_{t\,|t=\nu_s}} = \frac{1}{(\dot{v}_t + I\{t : \dot{v}_t = 0\})_{|t=\nu_s}} > 0. \tag{1.15}$$

Furthermore, for each $s \in [0, T]$, ν_s is an $\mathcal{F}_t^y$-stopping time and, as a process, (ν_s) is adapted to $\mathcal{F}_{\nu_s}^y = \sigma\{y_\tau : 0 \leq \tau \leq \nu_s\}$. Consider now the time-transformed observation process

$$\bar{y}_s = y_{\nu_s}, \tag{1.16}$$

which by (1.1) satisfies

$$d\bar{y}_s = A_{\nu_s}(\theta) x_{\nu_s} dv_{\nu_s} + d\eta_{\nu_s}. \tag{1.17}$$

Note that (η_{ν_s}) is a continuous, $\mathcal{F}_{\nu_s}^y$–conditionally Gaussian martingale with quadratic variation $\langle \eta \rangle_{\nu_s} = B^2 v_{\nu_s}$ such that

$$\frac{d}{ds}\langle \eta \rangle_{\nu_s} = \frac{B^2 \dot{v}_{\nu_s}}{\dot{v}_{\nu_s} + I\{s : \dot{v}_{\nu_s} = 0\}} = B^2 I\{s : \dot{v}_{\nu_s} \neq 0\}. \tag{1.18}$$

The process (η_{ν_s}) may thus degenerate and so, using a regularization procedure, we define

$$\tilde{\eta}_s := \eta_{\nu_s} + B \int_0^s I\{\tau : \dot{v}_{\nu_\tau} = 0\}\, dw_\tau, \tag{1.19}$$

where (w_t) is an $\mathcal{F}_t$-standard Wiener process, independent of $(w_t^{(1)})$ and (η_t). This process $(\tilde{\eta}_t)$ is thus a conditionally Gaussian martingale with respect to $\mathcal{F}^y_{\nu_s} = \mathcal{F}^{\bar{y}}_s$, that has continuous trajectories and nondegenerate quadratic variation

$$\begin{aligned}\langle\tilde{\eta}\rangle_s &= \langle\eta\rangle_{\nu_s} + B^2\int_0^s I\{\tau : \dot{v}_{\nu_\tau} = 0\}\, d\tau \\ &= B^2\int_0^s I\{\tau : \dot{v}_{\nu_\tau} \neq 0\}\, d\tau + B^2\int_0^s I\{\tau : \dot{v}_{\nu_\tau} = 0\}\, d\tau = B^2 s.\end{aligned} \tag{1.20}$$

Since $(\tilde{\eta}_s)$ is independent of $(w_s^{(1)})$, in what follows we shall consider an $\mathcal{F}_t$-Wiener process $(\tilde{w}_s^{(2)})$, independent of $(w_s^{(1)})$, and represent $(\tilde{\eta}_s)$ as $\tilde{\eta}_s = B\,\tilde{w}_s^{(2)}$. On the other hand, since for the process $(w_{\nu_s}^{(1)})$ we have $\langle w^{(1)}\rangle_{\nu_s} = \nu_s$, we may also consider an $\mathcal{F}_s$-standard Wiener process $(\tilde{w}_s^{(1)})$, independent of $(\tilde{w}_s^{(2)})$, and obtain $(w_{\nu_s}^{(1)})$ as

$$w_{\nu_s}^{(1)} = \int_0^s (\dot{\nu}_\tau)^{1/2} d\tilde{w}_\tau^{(1)}. \tag{1.21}$$

Defining finally for $s \le \Gamma_T$ (see (1.15))

$$\tilde{a}_s(\theta) := \frac{a_{\nu_s}(\theta)}{(\dot{v}_t + I\{t : \dot{v}_t = 0\})_{|t=\nu_s}} = a_{\nu_s}(\theta)\,\dot{\nu}_s, \tag{1.22a}$$

$$\tilde{b}_s(\theta) := b_{\nu_s}(\theta), \tag{1.22b}$$

$$\tilde{A}_s(\theta) := \frac{A_{\nu_s}(\theta)\,\dot{v}_{t\,|t=\nu_s}}{(\dot{v}_t + I\{t : \dot{v}_t = 0\})_{|t=\nu_s}} = A_{\nu_s}(\theta)\, I\{s : \dot{v}_{\nu_s} \neq 0\}, \tag{1.22c}$$

consider on $[0, T+M]$ the process-pair $(\tilde{x}_s, \tilde{y}_s)$ defined, for $0 \le s \le \Gamma_T$, by

$$d\tilde{x}_s = \tilde{a}_s(\theta)\tilde{x}_s ds + \tilde{b}_s(\theta)(\dot{\nu}_s)^{1/2} d\tilde{w}_s^{(1)}, \quad \tilde{x}_0 = x_0, \tag{1.23a}$$

$$d\tilde{y}_s = \tilde{A}_s(\theta)\tilde{x}_s ds + B\, d\tilde{w}_s^{(2)}, \quad \tilde{y}_0 = 0, \tag{1.23b}$$

$$d\theta = 0 \tag{1.23c}$$

and by putting $d\tilde{x}_s = d\tilde{y}_s = d\theta = 0$ for $\Gamma_T < s < T+M$. Note that from the foregoing we immediately have

$$\tilde{x}_s := x_{\nu_s}, \tag{1.24}$$

$$\bar{y}_s = \int_0^s I\{\tau : \dot{v}_{\nu_\tau} \neq 0\}\, d\tilde{y}_\tau \tag{1.25}$$

so that $\bar{y}_s$ defined in (1.16) is $\mathcal{F}^{\tilde{y}}_s$-measurable and therefore

$$\mathcal{F}^y_{\nu_s} = \mathcal{F}^{\bar{y}}_s \subseteq \mathcal{F}^{\tilde{y}}_s. \tag{1.26}$$

Analogously to (1.7) now consider

$$\tilde{m}_s^i := E\{\tilde{x}_s |\, \mathcal{F}^{\tilde{y}}_s, \theta^i\}, \tag{1.27a}$$

(1.27b) $$\tilde{\gamma}_s^i := E\{(\tilde{x}_s - \tilde{m}_s^i)^2 |\, \mathcal{F}_s^{\tilde{y}}, \theta^i\},$$

(1.27c) $$\tilde{\pi}_s^i := P\{\theta = \theta^i |\, \mathcal{F}_s^{\tilde{y}}\}.$$

We have the following proposition.

PROPOSITION 1.3. *The process* $\tilde{X}_s := \{\tilde{m}_s^i, \tilde{\gamma}_s^i, \tilde{\pi}_s^i\}_{i=1,\dots,k}$ *satisfies on* $[0,T]$ *(for a given control)*

(1.28a) $$d\tilde{m}_s^i = \tilde{a}_s(\theta)\tilde{m}_s^i ds + B^{-2}\tilde{A}_s(\theta^i)\tilde{\gamma}_s^i\, [d\tilde{y}_s - \tilde{A}_s(\theta^i)\tilde{m}_s^i ds], \quad \tilde{m}_0^i = m_0^i = m_0,$$

(1.28b) $$d\tilde{\gamma}_s^i = 2\tilde{a}_s(\theta^i)\tilde{\gamma}_s^i ds + \tilde{b}_s^2(\theta^i)\, \dot{\nu}_s ds - B^{-2}[\tilde{A}_s(\theta^i)\tilde{\gamma}_s^i]^2 ds, \quad \tilde{\gamma}_0^i = \gamma_0^i\sigma_0,$$

(1.28c) $$d\tilde{\pi}_s^i = \pi_s^i \left[\tilde{A}_s(\theta^i)\tilde{m}_s^i - \sum_{j=1}^k \tilde{\pi}_s^j \tilde{A}_s(\theta^j)\tilde{m}_s^j\right] B^{-2} d\tilde{\xi}_t, \quad \tilde{\pi}_0^i = \pi_0^i = p_i,$$

where

(1.29) $$\tilde{\xi}_s := \int_0^s \left[dy_\tau - \sum_{j=1}^k \tilde{\pi}_\tau^j \tilde{A}_\tau(\theta^j)\tilde{m}_\tau^j d\tau\right].$$

Remark 1.4. From (1.22c) and (1.25) we have

(1.30) $$\int_0^s \tilde{A}_\tau(\theta^i)\tilde{m}_\tau^i d\tilde{y}_\tau = \int_0^s A_{\nu_\tau}(\theta^i)\tilde{m}_\tau^i d\bar{y}_\tau,$$

(1.31) $$\int_0^s \tilde{A}_\tau(\theta^i)\tilde{\gamma}_\tau^i d\tilde{y}_\tau = \int_0^s A_{\nu_\tau}(\theta^i)\tilde{\gamma}_\tau^i d\bar{y}_\tau$$

so that, besides being $\mathcal{F}_s^{\tilde{y}}$-adapted, the process $\tilde{X}_s$ in Proposition 1.3 can also be considered $\mathcal{F}_s^{\bar{y}}$-adapted. Since

(1.32) $$E\left\{\tilde{x}_s|\mathcal{F}_s^{\tilde{y}}\right\} = E\left\{E\left\{\tilde{x}_s|\mathcal{F}_s^{\tilde{y}}, \theta\right\}|\mathcal{F}_s^{\tilde{y}}\right\} = \sum_{i=1}^k \tilde{m}_s^i \tilde{\pi}_s^i,$$

it thus follows that (see (1.26))

(1.33) $$E\left\{\tilde{x}_s|\mathcal{F}_s^{\bar{y}}\right\} = E\left\{E\left\{\tilde{x}_s|\mathcal{F}_s^{\tilde{y}}\right\}|\mathcal{F}_s^{\bar{y}}\right\} = E\left\{\tilde{x}_s|\mathcal{F}_s^{\tilde{y}}\right\}.$$

Proof of Proposition 1.3. Note that the partially observed system $(\tilde{x}_s, \tilde{y}_s)$ defined in (1.23) is nondegenerate and corresponds to the so-called "conditionally Gaussian" case. For the first two sets of components in (1.28) we may thus make use of Theorem 12.1 in [8], whose assumptions can easily be seen to be satisfied; in fact (see (1.22a))

(1.34) $$\begin{aligned}\int_0^{T+M} |\tilde{a}_s(\theta)|\, I\{s : s \le \Gamma_T\} ds &= \int_0^{T+M} |a_{\nu_s}(\theta)|\, \dot{\nu}_s\, I\{s : s \le \Gamma_T\} ds \\ &= \int_0^T |a_s(\theta)| ds < +\infty,\end{aligned}$$

and analogously (see (1.22c))

$$\begin{aligned}\int_0^{T+M} \tilde{A}_s^2(\theta)\, I\{s : s \le \Gamma_T\} ds &= \int_0^{T+M} A^2_{\nu_s}(\theta)\, I\{s : s \le \Gamma_T\} dv_{\nu_s} \\ &= \int_0^T A_s^2(\theta) dv_s < +\infty.\end{aligned} \tag{1.35}$$

Furthermore, the function $\tilde{b}_s(\theta)$ is continuous and $(\dot{\nu}_s)^{1/2}$ is integrable.

For the components $(\tilde{\pi}_t^i)$ in (1.28) we make use of the general (innovations form) nonlinear filtering equation of Theorem 8.1 in [8], putting, for a generic $i \le k$ and all $t \ge 0$,

$$h_s = h_s(\theta) := I\{\theta = \theta^i\}. \tag{1.36}$$

The assumptions of Theorem 8.1 in [8] are satisfied, and equation (8.10) in [8] with $H = D = 0$ then leads to

$$\begin{aligned} d\tilde{\pi}_s^i = B^{-2} &\left[\pi_s\left(I\{\theta = \theta^i\}\tilde{A}_s(\theta)\tilde{x}_s\right) - \pi_s\left(I\{\theta = \theta^i\}\right)\pi_s\left(\tilde{A}_s(\theta)\tilde{x}_s\right)\right] \\ \times &\left[d\tilde{y}_s - \pi_s\left(\tilde{A}_s(\theta)\tilde{x}_s\right) ds\right],\end{aligned} \tag{1.37}$$

where $\pi_s(Z) := E(Z|\mathcal{F}_s^{\tilde{y}})$. Noting now that

$$\begin{aligned}&\pi_s\left(I\{\theta = \theta^i\}\tilde{A}_s(\theta)\tilde{x}_s\right) \\ &\quad = E\left\{E\left\{I\{\theta = \theta^i\}\tilde{A}_s(\theta)\tilde{x}_s|\mathcal{F}_s^{\tilde{y}}, \theta\right\}|\mathcal{F}_s^{\tilde{y}}\right\} = \tilde{A}_s(\theta^i)\tilde{m}_s^i\,\tilde{\pi}_s^i\end{aligned} \tag{1.38}$$

and analogously

$$\pi_s\left(\tilde{A}_s(\theta)\tilde{x}_s\right) = \sum_{j=1}^k \tilde{A}_s(\theta^j)\tilde{m}_s^j\,\tilde{\pi}_s^j, \tag{1.39}$$

it follows that (1.37) is exactly (1.28c). We are now in a position to come to the proof of Theorem 1.2.

Proof of Theorem 1.2. Note first that, by (1.16), (1.24), and (1.33),

$$m_t^i = E\left\{\tilde{x}_{\Gamma(t)}|\mathcal{F}^{\tilde{y}}_{\Gamma(t)}, \theta^i\right\} = E\left\{\tilde{x}_{\Gamma(t)}|\mathcal{F}^{\tilde{y}}_{\Gamma(t)}, \theta^i\right\} = \tilde{m}^i_{\Gamma(t)}. \tag{1.40}$$

Analogously,

$$\gamma_t^i = \tilde{\gamma}^i_{\Gamma(t)}, \qquad \pi_t^i = \tilde{\pi}^i_{\Gamma(t)}. \tag{1.41}$$

From (1.40) and (1.28a) we obtain

$$m_t^i = \tilde{m}^i_{\Gamma(t)} = \tilde{m}_0^i + \int_0^{\Gamma(t)} \tilde{a}_s(\theta^i)\tilde{m}_s^i\, ds + B^{-2}\int_0^{\Gamma(t)} \tilde{A}_s(\theta^i)\tilde{\gamma}_s^i\,[d\tilde{y}_s - \tilde{A}_s(\theta^i)\tilde{m}_s^i\, ds]. \tag{1.42}$$

We now evaluate the integrals in this last expression, namely (see (1.22) and (1.16) with (1.25)),

$$\int_0^{\Gamma(t)} \tilde{a}_s(\theta^i)\tilde{m}_s^i\, ds = \int_0^{\Gamma(t)} a_{\nu_s}(\theta^i)\tilde{m}_s^i\, \dot{\nu}_s ds = \int_0^t a_\tau(\theta^i) m_\tau^i\, d\tau, \tag{1.43}$$

$$
\begin{aligned}
&\int_0^{\Gamma(t)} \tilde{A}_s(\theta^i)\tilde{\gamma}^i_s\,[d\tilde{y}_s - \tilde{A}_s(\theta^i)\tilde{m}^i_s\,ds] \\
&\qquad = \int_0^{\Gamma(t)} A_{\nu_s}(\theta^i)\tilde{\gamma}^i_s\,[A_{\nu_s}(\theta)\dot{v}_{\nu_s}\dot{\nu}_s\tilde{x}_s\,ds + d\eta_{\nu_s} - A_{\nu_s}(\theta^i)\dot{v}_{\nu_s}\dot{\nu}_s\tilde{m}^i_s\,ds] \\
&\qquad = \int_0^t A_\tau(\theta^i)\gamma^i_\tau\,[A_\tau(\theta)x_\tau\,dv_\tau + d\eta_\tau - A_\tau(\theta^i)m^i_\tau\,dv_\tau] \\
&\qquad = \int_0^t A_\tau(\theta^i)\gamma^i_\tau\,[dy - A_\tau(\theta^i)m^i_\tau\,dv_\tau].
\end{aligned} \tag{1.44}
$$

Substituting (1.43) and (1.44) into (1.42) we obtain (1.9a). The remaining equations for (γ^i_t) and (π^i_t) in (1.9) follow analogously.

Coming to the statement of Theorem 1.2 concerning the process (ξ_t) note that, according to Theorem 7.12 in [8], the process $(\tilde{y}_s)$ defined in (1.23) admits the representation

$$\tilde{y}_s = \int_0^s E\left\{\tilde{A}_\tau(\theta)\tilde{x}_\tau|\mathcal{F}^{\tilde{y}}_\tau\right\}d\tau + \tilde{w}_s = \int_0^s \sum_{j=1}^k \tilde{A}_\tau(\theta^j)\,\tilde{m}^j_\tau\tilde{\pi}^j_\tau\,d\tau + \tilde{w}_s, \tag{1.45}$$

where $(\tilde{w}_s)$ is an $\mathcal{F}^{\tilde{y}}_s$-standard Wiener process and the second equality follows from (1.39). As a consequence

$$\tilde{w}_s = \int_0^s \left[d\tilde{y}_s - \sum_{j=1}^k \tilde{A}_\tau(\theta^j)\,\tilde{m}^j_\tau\tilde{\pi}^j_\tau\,d\tau\right]. \tag{1.46}$$

On the other hand, from the definition of (ξ_t) in (1.10) and from (1.16), (1.15), (1.22), and (1.25), it then follows that

$$
\begin{aligned}
\xi_t &= \int_0^t \left[dy_s - \sum_{j=1}^k A_s(\theta^j)\,m^j_s\pi^j_s\,dv_s\right] \\
&= \int_0^{\Gamma(t)} \left[d\bar{y}_s - \sum_{j=1}^k \tilde{A}_{\nu_s}(\theta^j)\,\tilde{m}^j_s\tilde{\pi}^j_s\,I\{s:\dot{v}_{\nu_s}\neq 0\}ds\right] \\
&= \int_0^{\Gamma(t)} I\{s:\dot{v}_{\nu_s}\neq 0\}\left[d\tilde{y}_s - \sum_{j=1}^k \tilde{A}_{\nu_s}(\theta^j)\,\tilde{m}^j_s\tilde{\pi}^j_s\,ds\right] \\
&= \int_0^{\Gamma(t)} I\{s:\dot{v}_{\nu_s}\neq 0\}\,d\tilde{w}_s,
\end{aligned} \tag{1.47}
$$

from which

$$\langle\xi\rangle_t = \int_0^{\Gamma(t)} I\{s:\dot{v}_{\nu_s}\neq 0\}ds = \int_0^{\Gamma(t)} dv_{\nu_s} = v_{\nu_{\Gamma(t)}} = v(t). \tag{1.48}$$

It follows that there exists an $\mathcal{F}_t$-Wiener process (w^x_t) such that

$$d\xi_t = u_t^{1/2}dw^x_t. \tag{1.49}$$

On the other hand, the driving random process in (1.9a) can, using (1.10) and (1.49), be expressed as

$$
\begin{aligned}
&[dy_t - A_t(\theta^i) m_t^i dv_t] \\
&= \left[dy_t - \sum_{j=1}^{k} \pi_t^j A_t(\theta^j) m_t^j dv_t \right] + \left[\sum_{j=1}^{k} \pi_t^j A_t(\theta^j) m_t^j - A_t(\theta^i) m_t^i \right] dv_t \\
&= u_t^{1/2} dw_t^x + \left[\sum_{j=1}^{k} \pi_t^j A_t(\theta^j) m_t^j - A_t(\theta^i) m_t^i \right] u_t dt.
\end{aligned} \tag{1.50}
$$

Equation (1.12) now follows from (1.9), thus concluding the first part of the proof of the theorem.

Concerning the uniqueness, we start from the equation (1.9b) for (γ_t^i). Its solution is uniformly bounded, implying the local Lipschitzianity (with integrable Lipschitz constant) of the right-hand side in (1.9b). Coming to (1.9a) for (m_t^i), its uniqueness follows from the linearity in m_t^i of the right-hand side. Finally concerning the equation for (π_t^i), consider the auxiliary process $(\pi_{t\wedge t_n}^i)$, where

$$
t_n := \inf\{t : \max_i |m_t^i| = n\} \wedge T, \tag{1.51}
$$

for which it is easily seen that $t_n \uparrow T$. Due to the linearity of its right-hand side, the equation for $(\pi_{t\wedge t_n}^i)$ now admits a unique solution that coincides with (π_t^i) for $t \le t_n$. If, besides (π_t^i), there is also a solution $(\hat{\pi}_t^i)$, then $\pi_{t\wedge t_n}^i = \hat{\pi}_{t\wedge t_n}^i$ so that, by $t_n \uparrow T$ and the continuity of π_t^i, it follows that $\pi_t^i = \hat{\pi}_t^i \ \ \forall t \in [0, T]$.

2. The control problem.

2.1. Formulation of the control problem. The purpose of the control problem is to choose the control v_t, or equivalently u_t, in (1.1) to maximize the information content of the observations regarding the state. This information content can be measured by the precision of the estimation of x_t on the basis of the observation history $y_0^t := \{y_s;\, 0 \le s \le t\}$, which is given by the inverse of the conditional estimation covariance

$$
\begin{aligned}
\gamma_t = \mathrm{Cov}(x_t | \mathcal{F}_t^y) &= E\left\{ \left(x_t - \sum_{i=1}^{k} \pi_t^i m_t^i \right)^2 | \mathcal{F}_t^y \right\} \\
&= \sum_{i=1}^{k} \pi_t^i \left(\gamma_t^i + (m_t^i)^2 \right) - \left(\sum_{i=1}^{k} \pi_t^i m_t^i \right)^2 .
\end{aligned} \tag{2.1}
$$

The control objective can therefore be seen as minimizing γ_t for each t. More generally, taking into consideration also a possible penalization of the control effort, we shall consider as control objective the minimization of the following (finite-horizon) functional:

$$
J(u) = E\left\{ \int_0^T \left[f_t^0(\gamma_t) + f_t^1(\gamma_t) u_t \right] dt + \phi^0(\gamma_T) \right\}, \tag{2.2}
$$

where f^0, f^1, and ϕ^0 are continuous functions of polynomial growth in γ. Note that the filter components γ_t^i and π_t^i are uniformly bounded; on the other hand, by (1.10), equation (1.9a) can (see also the proof of Theorem 1.2) be rewritten as

(2.3)
$$dm_t^i = a_t(\theta^i)\, m_t^i\, dt + B^{-2}A_t(\theta^i)\gamma_t^i \left[\sum_{j=1}^{k} \pi_t^j A_t(\theta^j)m_t^j - A_t(\theta^i)m_t^i\right] u_t dt + B^{-2}A_t(\theta^i)\gamma_t^i d\xi_t,$$

from which it follows that, for a given control u_t, m_t^i possesses uniformly bounded moments of all orders. By (2.1) and the polynomial growth property of f^0, we thus have the existence of the expectation

(2.4)
$$E\{f_t^0(\gamma_t)\} = E\{E\{f_t^0(\gamma_t)|\mathcal{F}_t^y\}\} = E\left\{f_t^0\left(\sum_{i=1}^{k}\pi_t^i\,(\gamma_t^i + (m_t^i)^2) - \left(\sum_{i=1}^{k}\pi_t^i m_t^i\right)^2\right)\right\} := E\{F_t^0(X_t)\},$$

with X_t as in (1.8). Analogously, for the remaining two terms in (2.2) we have

$$E\left\{f_t^1(\gamma_t)u_t\right\} = E\left\{F_t^1(X_t)\, u_t\right\}, \tag{2.5}$$

$$E\left\{\phi^0(\gamma_T)\right\} = E\left\{\Phi^0(X_T)\right\}. \tag{2.6}$$

From (2.4)–(2.6), which implicitly define the functions F^0, F^1, and Φ^0, we have that the criterion function in (2.2) is well defined for any control (u_t) satisfying (1.2) and that $J(u)$ can equivalently be represented as

$$J'(u) = E\left\{\int_0^T \left[F_t^0(X_t) + F_t^1(X_t)u_t\right] dt + \Phi^0(X_T)\right\} \tag{2.7}$$

for suitable functions F^0, F^1, and Φ^0 that inherit the polynomial growth property of f^0, f^1, and ϕ^0 (the prime distinguishing the representation (2.3) from that in (2.2)).

2.2. The separation property. So far the admissible controls were assumed to be $\mathcal{F}_t^y$-adapted. In line with stochastic control under partial state observation one may investigate whether among the possible $\mathcal{F}_t^y$-adapted optimal controls there is one that is $\mathcal{F}_t^X$- adapted, namely, a function of the observations through the filter values. This is in fact so, and for this purpose consider the two classes of controls

(2.8)
$$\mathcal{L}_0 := \{u : u_t \text{ is } \mathcal{F}_t^y\text{-adapted, is Lipschitz in the sense of (1.5), and satisfies (1.2)}\},$$

$$\mathcal{L}_1 := \{u \in \mathcal{L}_0 : u_t \text{ is in particular } \mathcal{F}_t^X\text{-adapted}\}. \tag{2.9}$$

Given these two classes of controls and recalling (2.2) and (2.7), we have the following *separation theorem*, which allows us to consider, instead of the original control system (1.1) with admissible controls in $\mathcal{L}_0$ and criterion functional $J(u)$ according to (2.2), the equivalent problem for the filter system (1.12), admissible controls in $\mathcal{L}_1$, and criterion functional $J'(u)$ (see (2.7).

THEOREM 2.1. *Let* $\gamma_0 = \text{Cov}(x_0) > 0$ *and* $f_t^1(\gamma_t) \geq 0$. *Then the strong principle of separation holds, namely,*

$$\inf_{u \in \mathcal{L}_1} J'(u) = \inf_{u \in \mathcal{L}_0} J(u). \tag{2.10}$$

Proof. Since $\mathcal{L}_1 \subseteq \mathcal{L}_0$, we immediately have

$$\inf_{u \in \mathcal{L}_1} J'(u) \geq \inf_{u \in \mathcal{L}_0} J(u), \tag{2.11}$$

so we need to show only the opposite inequality. For this purpose, defining

$$N_A := \left\{ t : \sup_{i \leq k} |A_t(\theta^i)| = 0 \right\}, \tag{2.12}$$

consider the subclasses of controls

$$\bar{\mathcal{L}}_0 = \{u \in \mathcal{L}_0 : u_t = 0 \quad \text{for} \quad t \in N_A\}, \tag{2.13a}$$

$$\bar{\mathcal{L}}_1 = \{u \in \mathcal{L}_1 : u_t = 0 \quad \text{for} \quad t \in N_A\}. \tag{2.13b}$$

For $u \in \mathcal{L}_0$ let

$$\bar{u}_t := u_t \, I\{t \notin N_A\} \in \bar{\mathcal{L}}_0, \tag{2.14}$$

and we have

$$J(\bar{u}) \leq J(u). \tag{2.15}$$

Analogously for $u \in \mathcal{L}_1$. In fact, it is easily seen from (1.9) that $u(\cdot)$ and $\bar{u}(\cdot)$ generate the same process $X_t = \left\{ m_t^i, \gamma_t^i, \pi_t^i \right\}_{i=1,\ldots,k}$, while due to the nonnegativity of $f_t^1(\gamma)$ and of u_t, one has

$$\int_0^T f_t^1(\gamma_t) \bar{u}_t \, dt \leq \int_0^T f_t^1(\gamma_t) u_t \, dt. \tag{2.16}$$

It follows that

$$\inf_{u \in \bar{\mathcal{L}}_0} J(u) \leq \inf_{u \in \mathcal{L}_0} J(u), \tag{2.17}$$

and, since $\bar{\mathcal{L}}_0 \subseteq \mathcal{L}_0$, we have

$$\inf_{u \in \bar{\mathcal{L}}_0} J(u) = \inf_{u \in \mathcal{L}_0} J(u) \tag{2.18}$$

and, analogously,

$$\inf_{u \in \bar{\mathcal{L}}_1} J'(u) = \inf_{u \in \mathcal{L}_1} J'(u). \tag{2.19}$$

Now let $\bar{u}$ denote a control with $\bar{u}_t = 0$ for $t \in N_A$; in particular, we may think of $\bar{u} \in \bar{\mathcal{L}}_0$. Corresponding to any such given control $\bar{u}$, the σ-algebra $\mathcal{F}_t^y$ generated by

y_s for $0 \le s \le t$ is contained in that $\mathcal{F}_t^X$ generated by $X_s = \{m_s^i, \gamma_s^i, \pi_s^i\}_{i=1,\dots,k}$ for $0 \le s \le t$. In fact, since $\gamma_0 > 0$, passing to its inverse, we have from (1.9b) that $\gamma_t > 0$ for all $t \in [0,T]$. Furthermore, on the complement of N_A there exists at least one value of $i \in \{1,\dots,k\}$ for which $A_t(\theta^i)\gamma_t^i \neq 0$; consequently, taking into account the continuity of $A_t(\theta^i)\gamma_t^i$, one can choose a measurable function $i(t)$ with values in $\{1,\dots,k\}$ so that

$$A_t(\theta^{i(t)})\gamma_t^{i(t)} \neq 0 \qquad \text{for} \quad t \in \bar{N}_A, \tag{2.20}$$

where $\bar{N}_A$ denotes the complement of N_A. This set $\bar{N}_A$ can then be represented in the form $\bar{N}_A = \cup_{i=1}^k \bar{N}_i$, where

$$\bar{N}_i := \{t : i(t) = i\}, \quad i = 1,\dots,k, \tag{2.21}$$

with $A_t(\theta^i)\gamma_t^i \neq 0$ for $t \in \bar{N}_i$, and we have

$$I\{t \notin N_A\} = \sum_i I\{t \notin N_i\}. \tag{2.22}$$

Recalling then that $\bar{u}_t = 0$ for $t \in N_A$, for the observation process we have

$$y_t = \int_0^t I\{s \notin N_A\}\, dy_s = \sum_{i=1}^k \int_0^t I\{s \notin N_i\}\, dy_s. \tag{2.23}$$

On the other hand, from (1.9a) we obtain

$$\begin{aligned} &\int_0^t B^{-2} A_s(\theta^i)\gamma_s^i\, dy_s \\ &\quad = \int_0^t \left(dm_s^i - a_s(\theta^i) m_s^i\, ds\right) + \int_0^t B^{-2} A_s^2(\theta^i)\gamma_s^i\, m_s^i\, \bar{u}_s\, ds. \end{aligned} \tag{2.24}$$

Multiplying the integrands by $\left(B^{-2}A_s(\theta^i)\gamma_s^i\right)^+ I\{s \notin N_i\}$, where $(\cdot)^+$ denotes the generalized inverse, it follows that

$$\begin{aligned} \int_0^t I\{s \notin N_i\}\, dy_s = &\int_0^t \left(B^{-2}A_s(\theta^i)\gamma_s^i\right)^+ I\{s \notin N_i\} \left[dm_s^i - a_s(\theta^i) m_s^i ds\right] \\ &+ \int_0^t I\{s \notin N_i\} A_s(\theta^i)\, m_s^i\, \bar{u}_s\, ds, \end{aligned} \tag{2.25}$$

from which, taking (2.23) into account, we obtain for y_t the following representation:

$$\begin{aligned} y_t = \sum_{i=1}^k \int_0^t I\{s \notin N_i\} \Big\{ &\left(B^{-2}A_s(\theta^i)\gamma_s^i\right)^+ \left[dm_s^i - a_s(\theta^i) m_s^i ds\right] \\ &+ A_s(\theta^i)\, m_s^i\, \bar{u}_s\, ds \Big\}. \end{aligned} \tag{2.26}$$

This representation shows that, given a control $\bar{u}$ with $\bar{u}_t = 0$ for $t \in N_A$, the process (y_t) is also $\mathcal{F}_t^X$-adapted and so $\mathcal{F}_t^y \subseteq \mathcal{F}_t^X$. As a consequence, we have that $\bar{\mathcal{L}}_0 \subseteq \bar{\mathcal{L}}_1$ so that by (2.18) and (2.19)

$$\inf_{u \in \mathcal{L}_1} J'(u) = \inf_{u \in \bar{\mathcal{L}}_1} J'(u) \le \inf_{u \in \bar{\mathcal{L}}_0} J(u) = \inf_{u \in \mathcal{L}_0} J(u), \tag{2.27}$$

which is the desired opposite inequality.

3. The auxiliary control problem and nearly optimal Lipschitz Markov controls. Our original control problem now consists of controlling the filter process

$$X_t = \{m_t^i, \gamma_t^i, \pi_t^i\}_{i=1,\dots,k}$$

evolving according to (1.12) in order to minimize (see (2.7))

$$J'(u) = E\left\{\int_0^T \left[F_t^0(X_t) + F_t^1(X_t)u_t\right] dt + \Phi^0(X_T)\right\}. \tag{3.1}$$

As follows from Theorem 2.1, we may limit ourselves to considering controls from the class $\mathcal{L}_1$ so that they satisfy also the constraints (1.2), i.e.,

$$\int_0^T u_t dt \le M < +\infty, \quad 0 \le u_t < +\infty. \tag{3.2}$$

It is a nonlinear control problem with unbounded controls, so an optimal solution may not exist. Using the so-called method of discontinuous time transformation (see [10] in a deterministic context, where it is used for the representation of generalized—in particular, discontinuous—solutions in problems with impulse control) next we transform this original problem into an auxiliary problem with bounded controls for which an optimal solution can be shown to exist.

3.1. Method of discontinuous time transformation. To describe the method, let $u \in \mathcal{L}_1$ and consider similarly to section 1.2 the function

$$\Gamma_t := t + \int_0^t u_s ds = t + v_t \tag{3.3}$$

as well as its inverse

$$\nu_s = \Gamma_s^{-1} = \inf\{\tau : \Gamma_\tau > s\}, \tag{3.4}$$

which is an $\mathcal{F}_{\nu_s}^X$-adapted process defined on $[0, \Gamma_T]$, where, due to (3.2), $\Gamma_T \le T + M$. Furthermore, it is absolutely continuous since, for any s_1, s_2 with $s_1 \le s_2$, we have

$$0 < \nu_{s_2} - \nu_{s_1} \le s_2 - s_1, \tag{3.5}$$

so it is almost everywhere differentiable on $[0, \Gamma_T]$ with derivative (see (3.3))

$$\dot{\nu}_s = \left[\dot{\Gamma}_t\right]^{-1}_{|t=\nu_s} = (1 + u_t)^{-1}_{|t=\nu_s}, \tag{3.6}$$

which is $\mathcal{F}_{\nu_s}^X$-adapted and satisfies

$$0 < \dot{\nu}_s \le 1. \tag{3.7}$$

Then considering the process-pair

$$Z_s := X_{\nu_s}, \qquad \mu_s := v_{\nu_s}, \tag{3.8}$$

where (see section 1.1)

$$v_t = \int_0^t u_\tau d\tau, \tag{3.9}$$

we have the following lemma.

LEMMA 3.1. *Let the process* (X_t) *satisfy* (1.12) *for some* $u \in \mathcal{L}_1$. *Then there exists an* $\mathcal{F}_{\nu_s}$*-Wiener process* (w_s^Z) *such that the process-triple* (Z_s, μ_s, ν_s) *satisfies, for* $s \in [0, \Gamma_T]$,

$$dZ_s = \alpha_s F_{\nu_s}(Z_s)ds + (1-\alpha_s)B_{\nu_s}(Z_s)ds + (1-\alpha_s)^{1/2}G_{\nu_s}(Z_s)dw_s^Z, \quad Z_0 = X_0, \tag{3.10a}$$

$$d\mu_s = (1-\alpha_s)\, ds, \quad \mu_0 = 0, \tag{3.10b}$$

$$d\nu_s = \alpha_s ds, \quad \nu_0 = 0, \tag{3.10c}$$

where the functions F, B, *and* G *are as in* (1.12), *the control* α *is given by*

$$\alpha_s = \dot{\nu}_s, \tag{3.11}$$

and it is $\mathcal{F}_s^Z$*-adapted and satisfies* $0 < \alpha_s \leq 1$. *Furthermore, the solution of* (3.10) *is unique.*

Proof. By (3.7) we have $0 < \alpha_s \leq 1$. By (1.12) the process $Z_s = X_{\nu_s}$ satisfies

$$Z_s = X_{\nu_s} = X_0 + \int_0^{\nu_s} F_t(X_t)dt + \int_0^{\nu_s} B_t(X_t)u_t dt + \int_0^{\nu_s} G_t(X_t)u_t^{1/2}dw_t^X. \tag{3.12}$$

Taking into account the identities

$$\nu_{\Gamma_t} = t, \quad \Gamma_{\nu_s} = s \tag{3.13}$$

valid for $t \in [0, T]$ and $s \in [0, \Gamma_T]$, we derive next a representation for the integrals in the right-hand side of (3.12), namely, performing the change of variables $t = \nu_\tau$,

$$\int_0^{\nu_s} F_t(X_t)dt = \int_0^{\Gamma_{\nu_s}} F_{\nu_\tau}(X_{\nu_\tau})d\nu_\tau = \int_0^s F_{\nu_\tau}(X_{\nu_\tau})\,\alpha_\tau d\tau, \tag{3.14}$$

$$\begin{aligned}\int_0^{\nu_s} B_t(X_t)u_t dt &= \int_0^{\nu_s} B_t(X_t)\frac{u_t}{1+u_t}(1+u_t)\, dt \\ &= \int_0^{\Gamma_{\nu_s}} B_{\nu_\tau}(X_{\nu_\tau})\left(1 - \frac{1}{1+u_t}\right)_{|t=\nu_\tau} d\Gamma_{\nu_\tau} \\ &= \int_0^s B_{\nu_\tau}(X_{\nu_\tau})\,(1-\alpha_\tau)\, d\tau,\end{aligned} \tag{3.15}$$

$$\int_0^{\nu_s} G_t(X_t)u_t^{1/2}dw_t^X = \int_0^{\Gamma_{\nu_s}} G_{\nu_\tau}(X_{\nu_\tau})\, u_{\nu_\tau}^{1/2}dw_{\nu_\tau}^X. \tag{3.16}$$

The process $w_{\nu_\tau}^X$ is an $\mathcal{F}_{\nu_\tau}^X$-adapted, conditionally Gaussian martingale with continuous trajectories and quadratic variation

$$\langle w^X \rangle_{\nu_\tau} = \nu_\tau = \int_0^\tau \alpha_u du. \tag{3.17}$$

There exists therefore an $\mathcal{F}_{\nu_s}$-Wiener process w_s^Z such that

$$w_{\nu_\tau}^X = \int_0^\tau (\alpha(u))^{1/2} dw_u^Z. \tag{3.18}$$

Substituting (3.18) into (3.16) we then obtain (see also (3.6) and (3.11))

$$\begin{aligned}\int_0^{\nu_s} G_t(X_t) u_t^{1/2} dw_t^X &= \int_0^s G_{\nu_\tau}(X_{\nu_\tau}) \left(\frac{u_{\nu_\tau}}{1+u_{\nu_\tau}}\right)^{1/2} dw_\tau^Z \\ &= \int_0^s G_{\nu_\tau}(X_{\nu_\tau}) (1-\alpha_\tau)^{1/2} dw_\tau^Z.\end{aligned} \tag{3.19}$$

Using (3.14), (3.15), and (3.19) in (3.12) we obtain (3.10a) for Z_s. Analogously, for the process μ_s we obtain

$$\begin{aligned}\mu_s = \int_0^{\nu_s} u_t dt &= \int_0^{\Gamma_{\nu_s}} u_{\nu_\tau} d\nu_\tau \\ &= \int_0^s \frac{u_{\nu_\tau}}{1+u_{\nu_\tau}} d\tau = \int_0^s (1-\alpha_\tau) d\tau.\end{aligned} \tag{3.20}$$

The uniqueness of a strong solution of (3.10) follows analogously to that of system (1.12) (see Theorem 1.2).

Next we shall establish a relationship converse to Lemma 3.1. Therefore consider (3.10) as a system controlled by a process α_s that is $\mathcal{F}_s^Z$-adapted and satisfies $0 < \alpha_s \le 1$ for $0 \le s \le S$, where S is an $\mathcal{F}_s^{Z,\mu}$-stopping time given by

$$S := S_\nu \wedge S_\mu, \tag{3.21}$$

with

$$S_\nu := \inf\{s : \nu_s = T\}, \tag{3.22a}$$

$$S_\mu := \inf\{s : \mu_s = M\}. \tag{3.22b}$$

By the fact that (see (3.10)) $\nu_s + \mu_s = s$, we have $S \le T + M$.

Define $\mathcal{A}$ as the class of $\mathcal{F}_s^Z$-adapted controls α satisfying $\alpha_s \in (0,1]$ for $s \in [0,S]$ and where, if $S = S_\mu$, we extend its definition, letting $\alpha_s = 1$ for $S_\mu < s \le T + M$. We then have the following lemma.

LEMMA 3.2. *Given* (3.10), *let* $\alpha \in \mathcal{A}$. *Then, putting* $\Gamma_t := \inf\{s : \nu_s > t\}$, *there exists an* $\mathcal{F}_{\Gamma_t}^{Z,\mu}$*-adapted control* (u_t) *satisfying* (1.2) (*see also* (3.2)) *and an* $\mathcal{F}_{\Gamma_t}^{Z,\mu}$*-Wiener process* (w_t^X) *such that, for* $t \le T$, *the processes*

$$X_t := Z_{\Gamma_t}, \quad v_t := \mu_{\Gamma_t} \tag{3.23}$$

satisfy (1.12) *with* $v_t = \int_0^t u_\tau d\tau$. *The control* (u_t) *is furthermore given by*

$$u_t = \dot{\Gamma}_t - 1 = \alpha_{\Gamma_t}^{-1} - 1 \tag{3.24}$$

and $\mathcal{F}_{\Gamma_t}^{Z,\mu} = \mathcal{F}_t^{X,v}$.

Proof. From its definition in (3.24), the control u_t is trivially $\mathcal{F}^{Z,\mu}_{\Gamma_t}$-adapted and satisfies (1.2a). Furthermore, under the assumptions of the lemma, we have

$$v_t = \int_0^t u_\tau d\tau = \int_0^t \left(\alpha_{\Gamma_t}^{-1} - 1\right) d\tau = \int_0^t \frac{1-\alpha_{\Gamma_\tau}}{\alpha_{\Gamma_\tau}} d\tau$$
$$= \int_0^t (1-\alpha_{\Gamma_\tau}) d\Gamma_\tau = \int_0^{\Gamma_\tau} (1-\alpha_s)\, ds = \mu_{\Gamma_\tau} \le M$$

so that (1.2b) also is satisfied and $v_t = \mu_{\Gamma_t}$. It remains to show that $X_t = Z_{\Gamma_t}$ satisfies (1.12). For this purpose note that, based on (3.14) and (3.15) as well as (3.24), we may write

$$\int_0^{\Gamma_t} F_{\nu_s}(Z_s)\, \alpha_s\, ds = \int_0^t F_\tau(Z_{\Gamma_\tau})\, d\tau, \tag{3.25}$$

$$\int_0^{\Gamma_t} B_{\nu_s}(Z_s)\, (1-\alpha_s)\, ds = \int_0^t B_\tau(Z_{\Gamma_\tau})\, u_\tau\, d\tau, \tag{3.26}$$

and, finally, based on (3.19)

$$\begin{aligned} \int_0^{\Gamma_t} G_{\nu_s}(Z_s)\, (1-\alpha_s)^{1/2}\, dw_s^Z &= \int_0^t G_\tau(Z_{\Gamma_\tau}) \left(\frac{u_\tau}{1+u_\tau}\right)^{1/2} dw^Z_{\Gamma_\tau} \\ &= \int_0^t G_\tau(Z_{\Gamma_\tau})\, u_\tau^{1/2} dw_\tau^X, \end{aligned} \tag{3.27}$$

where (see also (3.18) and (3.24))

$$w_t^X = \int_0^t \frac{dw^Z_{\Gamma_\tau}}{(1+u_\tau)^{1/2}} \tag{3.28}$$

is a continuous $\mathcal{F}^{Z,\mu}_{\Gamma_t} = \mathcal{F}^{X,v}_t$-martingale with quadratic variation

$$\langle w^X \rangle_t = \int_0^t \frac{d\Gamma_\tau}{1+u_\tau} = t,$$

and therefore an $\mathcal{F}^{X,v}_t$-Wiener process.

The results obtained in the two lemmas above allow us to consider, instead of the original controlled system (1.12) with unbounded controls, the system (3.10) where the controls are bounded. We are now going to define more precisely the control problem corresponding to this latter system, which we shall call the *auxiliary control problem.*

3.2. The auxiliary control problem. This auxiliary problem concerns the controlled system (Z_s, μ_s, ν_s) satisfying (3.10) with controls from an enlarged class $\mathcal{A}_0$ consisting of the controls in $\mathcal{A}$ (defined in Lemma 3.2), where we also allow the value $\alpha_s = 0$, i.e.,

$$\mathcal{A}_0 := \{\alpha \in \mathcal{A} \mid 0 \le \alpha_s \le 1\}. \tag{3.29}$$

This enlargement of the class of controls guarantees, as we shall see, the existence of an optimal solution for the auxiliary problem. On the other hand, through the correspondence (3.24), this is equivalent to allowing unbounded controls in the original problem.

As a cost functional to be minimized we consider

$$J(\alpha) = E\left\{\int_0^S \left[\alpha_s F^0_{\nu_s}(Z_s) + (1-\alpha_s)F^1_{\nu_s}(Z_s)\right] ds + \bar{\Phi}^0_{\nu_S}(Z_S)\right\}, \tag{3.30}$$

where S is the $\mathcal{F}^{Z,\mu}_s$-stopping time defined in (3.21) and (3.22), F^0 and F^1 are as in (3.1) or (2.7), and the terminal cost function is given by

$$\bar{\Phi}^0_\nu(Z) = \begin{cases} \Phi^0(Z) & \text{if } S = S_\nu, \\ \Phi^0(\psi_T(\nu, Z)) + \int_\nu^T F^0_s(\psi_s(\nu, Z))\, ds & \text{if } S = S_\mu, \end{cases} \tag{3.31}$$

with $\psi_s(\nu, Z)$ being the solution on $[\nu, T]$ of the deterministic equation

$$\dot{\psi}_s = F_s(\psi_s), \tag{3.32}$$

having initial condition $\psi_\nu(\nu, Z) = Z$. F is as in (1.12) (see also (3.10a)), and F^0, F^1, and Φ^0 are the same as in $J'(u)$ (see (3.1) or (2.7)).

Remark 3.3. The function $\psi_s(\nu, Z)$ satisfies $\psi_\nu(\nu, Z) = Z$, is continuous in all variables, and has linear growth with respect to Z, since the function $F_s(X)$ is continuous and Lipschitz in X for each s.

3.3. Relationship between the original and auxiliary problems. In this section we will show the correspondence existing between the cost functionals $J'(u)$ in (3.1) and $J(\alpha)$ in (3.30). We have in fact the following proposition.

PROPOSITION 3.4. *If $u \in \mathcal{L}_1$ is given and α is according to* (3.11) *and* (3.6)*, or $\alpha \in \mathcal{A}$ is given, $S = S_\nu$, and u is according to* (3.24)*, then*

$$J'(u) = J(\alpha). \tag{3.33}$$

When $\alpha \in \mathcal{A}$ but $S = S_\mu$ so that $\nu_S < T$, then (3.33) *continues to hold with u according to* (3.24) *if* (*see the definition of the class $\mathcal{A}$ before Lemma* 3.2) *one puts $\alpha_s = 1$ for $S_\mu < s \le T + M$.*

Proof. For the first part of the statement note the following: given a control $u \in \mathcal{L}_1$ and letting (see (3.11)) $\alpha_s = \dot{\nu}_s$, then since (see (3.8) and (3.9)) $\mu_s = \int_0^{\nu_s} u_\tau \, d\tau$ and $u \in \mathcal{L}_1$ satisfies (3.2), we have $S = S_\nu = \Gamma_T$ with $\Gamma(\cdot)$ as in (3.3) or, equivalently, as in the statement of Lemma 3.2. As a consequence we have

$$\nu_S = T, \quad Z_S = X_T, \tag{3.34}$$

and, by considerations analogous to those leading to (3.14) and (3.15), we then obtain

$$\int_0^T F^0_t(X_t)\, dt = \int_0^S F^0_{\nu_s}(Z_s)\, \alpha_s \, ds, \tag{3.35}$$

$$\int_0^T F^1_t(X_t)\, u_t \, dt = \int_0^S F^1_{\nu_s}(Z_s)\, (1-\alpha_s)\, ds. \tag{3.36}$$

On the other hand, given a control $\alpha \in \mathcal{A}$, if $S = S_\nu$, the same relations (3.34)–(3.36) hold. Combining these considerations with Lemmas 3.1 and 3.2 we obtain the first part of the proposition. The second part follows immediately, taking (3.31) into account.

Remark 3.5. The previous equivalence considerations are valid for $\alpha \in \mathcal{A}$, i.e., such that $\alpha_s > 0$. For the purpose of obtaining existence of an optimal solution for the auxiliary problem, we shall allow also controls in $\mathcal{A}_0$ (see (3.29)) so that α_s might be equal to zero on some subintervals of $[0, S]$. Correspondingly, on these subintervals, ν_s will be constant implying that its inverse $\Gamma_t = \inf\{s : \nu_s > t\}$ jumps. Consequently also $X_t = Z_{\Gamma_t}$ and $v_t = \mu_{\Gamma_t}$ will jump and can therefore not be a solution of (1.12) for any measurable control. In other words, while the auxiliary control problem admits an optimal solution, there may not exist a corresponding optimal solution for the original problem. We shall therefore determine nearly optimal (ε-optimal) solutions for the original problem.

Letting

$$\mathcal{A}_0^L := \{\alpha \in \mathcal{A}_0 : \alpha_t = \alpha_t(Z_t, \mu_t) \text{ a Lipschitz function}\} \tag{3.37}$$

and, analogously, for $\mathcal{A}^L$, we first prove the following.

PROPOSITION 3.6. *For any control $\alpha \in \mathcal{A}_0^L$ there exists a sequence of controls $\alpha^k \in \mathcal{A}^L$ obtained as*

$$\alpha_s^k = \frac{1}{(k+1)} + \frac{k}{(k+1)}\alpha_s, \tag{3.38}$$

where $s \in [0, S]$ if $S = S_\nu$ and $s \in [0, T + M]$ if $S = S_\mu$, such that

$$\lim_{k\to\infty} J(\alpha^k) = J(\alpha). \tag{3.39}$$

Proof. Given $\alpha \in \mathcal{A}_0^L$, let S be the corresponding stopping time defined according to (3.21) and (3.22). Define the sequence $\alpha^k \in \mathcal{A}^L$ as in (3.38). Also let

$$S^k := S_\nu^k \wedge S_\mu^k, \tag{3.40}$$

where S_ν^k and S_μ^k are defined according to (3.22) with $\nu = \nu^k$ and $\mu = \mu^k$ that correspond to α^k via (3.10). The sequence α^k is monotonically decreasing and converges to α.

Let us first show that the sequence S^k converges for all ω to S. In fact, since $\alpha_s^k \geq \alpha_s$ and

$$\alpha_s^{k+1} - \alpha_s^k = \frac{(\alpha_s - 1)}{(k+1)(k+2)} \leq 0$$

for the stopping times $S_\nu^k = \inf\left\{s : \nu_s^k = \int_0^s \alpha_\tau^k d\tau = T\right\}$, we have

$$S_\nu^k \leq S_\nu^{k+1} \leq S_\nu, \tag{3.41}$$

and analogously for $S_\mu^k = \inf\left\{s : \mu_s^k = \int_0^s (1 - \alpha_\tau^k)\, d\tau = M\right\}$ we get

$$S_\mu \leq S_\mu^{k+1} \leq S_\mu^k. \tag{3.42}$$

We can now prove that $\lim_{k\to\infty} S_\nu^k = S_\nu$. From (3.41) the limit of S_ν^k exists and, denoting it by $\bar{S}_\nu$, we have $\bar{S}_\nu \leq S_\nu$. By the uniform convergence of α^k to α we have that ν_s^k converges to ν_s uniformly on compact subintervals of $[0, S_\nu]$, where, we recall, $S_\nu \leq T + M$. In addition

$$\nu_{S_\nu^k}^k = T, \quad \nu_{S_\nu} = T.$$

Let us determine the value of $\nu_{\bar{S}_\nu}$. From

$$\nu^k_{S^k_\nu} - \nu_{\bar{S}_\nu} = \nu^k_{S^k_\nu} - \nu_{S^k_\nu} + \nu_{S^k_\nu} - \nu_{\bar{S}_\nu},$$

the uniform convergence of ν^k_s to ν_s, and the continuity of ν_s it follows that

$$\nu_{\bar{S}_\nu} = \lim_{k\to\infty} \nu^k_{S^k_\nu} = T. \tag{3.43}$$

Now suppose that $\bar{S}_\nu < S_\nu$ for some ω; then (3.43) contradicts the definition of S_ν as the stopping time according to (3.22), and consequently $\lim_{k\to\infty} S^k_\nu = S_\nu$. Analogously we obtain $\lim_{k\to\infty} S^k_\mu = S_\mu$ and finally

$$\lim_{k\to\infty} S^k = \lim_{k\to\infty} \left(S^k_\nu \wedge S^k_\mu\right) = S_\nu \wedge S_\mu = S.$$

Consider next the sequence (Z^k_s), with Z^k_s obtained as solutions of (3.10) corresponding to $\alpha = \alpha^k$. For each given $N > 0$ determine the sequence of stopping times $\theta^{N,k} = \min\{S, S^k, \tau^N, \tau^{N,k}\}, k = 1, 2, \ldots,$ where

$$\tau^N = \inf\{s : ||Z_s|| = N\}, \tag{3.44a}$$

$$\tau^{N,k} = \inf\{s : ||Z^k_s|| = N\}. \tag{3.44b}$$

The properties of the coefficients in the right-hand side of (3.10) guarantee that the trajectories of Z^k_s are a.s. continuous so that, for any $\alpha \in \mathcal{A}^L_0$ and any k,

$$\tau^N,\ \tau^{N,k} \uparrow \infty \qquad \text{a.s. as} \quad N \to \infty. \tag{3.45}$$

Consider next the sequences of processes $Z^k_{s\wedge\theta^{N,k}}$ as well as $Z_{s\wedge\theta^{N,k}}$. By the continuity and the local Lipschitzianity with respect to Z of the functions in the right-hand side of (3.10) as well the uniform convergence of $\nu^k_s \to \nu_s$ and of $\alpha^k_s \to \alpha_s$ on compact subsets of $[0, S_\nu]$, we obtain

$$\begin{aligned} \sup_{\tau\le s\wedge\theta^{N,k}} & E\,||\,Z^k_{\tau\wedge\theta^{N,k}} - Z_{\tau\wedge\theta^{N,k}}||^2 \\ & \le C_1 \int_0^{s\wedge\theta^{N,k}} \sup_{\tau\le u} E\,||\,Z^k_{\tau\wedge\theta^{N,k}} - Z_{\tau\wedge\theta^{N,k}}||^2 du + C_2\varepsilon_k, \end{aligned} \tag{3.46}$$

where $\lim_{k\to\infty} \varepsilon_k = 0$. Applying the Gronwall–Bellman inequality to (3.45) we get for all $s \le S$

$$\lim_{k\to\infty} \sup_{\tau\le s\wedge\theta^{N,k}} E\,||\,Z^k_\tau - Z_\tau||^2 = 0. \tag{3.47}$$

Together with k now also let $N \uparrow \infty$; then $\theta^{N,k} \to S$ a.s., implying that for all $s < S$ we have the convergence in $\mathcal{L}^2$ of Z^k_s to Z_s and, by the continuity of Z^k_s in $s = S$, also of $Z^k_{S^k}$ to Z_S.

This convergence in turn implies the convergence in probability of Z^k_s to Z_s for all $s \in [0, S]$ as well as that of $Z^k_{S^k}$ to Z_S. In addition we have the uniform integrability of Z^k_s on $\Omega \times [0, T + M]$ and of $Z^k_{S^k}$ on Ω since, by the linear growth in Z of the functions in the right-hand side of (3.10), we have for $p \ge 1$

$$\int_0^{T+M} E\,||Z^k_s||^p ds \le L < \infty, \quad E\,||Z^k_{S^k}||^p \le L < \infty. \tag{3.48}$$

As a consequence, and due to the polynomial growth in Z of the functions F^0, F^1, and $\bar{\Phi}^0$, we may pass to the limit in (3.30), thus concluding the proof of the proposition.

Combining the result of Proposition 3.6 with that of Proposition 3.4 we immediately obtain the following corollary.

COROLLARY 3.7. *For any control* $\alpha \in \mathcal{A}_0^L$ *there exists a sequence of Lipschitz Markov (feedback) controls* $u_t^k = u_t^k(X_t, v_t)$ *satisfying* (1.2) *such that*

$$\lim_{k\to\infty} J'(u^k) = J(\alpha). \tag{3.49}$$

Proof. From (3.38) we immediately have that, if $\alpha_t = \alpha_t(Z_t, \mu_t)$ with $\alpha_t(\cdot)$ Lipschitz, α^k also is Lipschitz. Recall next that the u^k corresponding to α^k is defined by (3.24) and (see Lemma 3.2) satisfies (1.2). Using the relationship (3.23) as well as the fact that $\alpha_s^k \geq \frac{1}{1+k}$, for such a u^k we then have $u_t^k = u_t^k(X_t, v_t)$ with $u_t^k(\cdot)$ Lipschitz. The result then follows by combining the previous considerations with Proposition 3.4.

3.4. Nearly optimal Lipschitz Markov controls. In this section we study first the existence of an optimal solution for the auxiliary control problem in the class $\mathcal{A}_0$ as well as the existence of a nearly optimal Lipschitz Markov control for the original problem. We then return to the relationship between the original and the auxiliary control problems, showing the usefulness of the auxiliary problem to obtain a nearly optimal Lipschitz Markov control in the original problem.

For the first part we have the following theorem.

THEOREM 3.8. *In the class* $\mathcal{A}_0$ *there exists an optimal control for the auxiliary problem, and it is of the Markov (feedback) type*

$$\alpha_s^0 = \alpha_s^0(Z_s, \mu_s). \tag{3.50}$$

Furthermore, for any $\varepsilon > 0$ *there exists a Lipschitz Markov control* $\alpha_s^{0,\varepsilon}(Z_s, \mu_s) \in \mathcal{A}_0$ *such that*

$$J(\alpha^0) = \inf_{\alpha\in\mathcal{A}_0} J(\alpha) \geq J(\alpha^{0,\varepsilon}) - \varepsilon. \tag{3.51}$$

Proof. Concerning the existence of an optimal solution in $\mathcal{A}_0$ note first that, under our assumptions, the set

$$K(\nu, Z) = \left\{ \begin{matrix} \alpha F + (1-\alpha)B, & 1-\alpha, & \alpha, & (1-\alpha)GG^T \\ \alpha F^0 + (1-\alpha)F^1 & & & \end{matrix} \right\}_{|\alpha\in[0,1]} \tag{3.52}$$

is, for all (ν, Z), bounded, closed, and convex. This allows us to apply known results on the existence of optimal controls—in particular, Theorem 5.15 in [3]—since the functions F, B, GG^T, F^0, and F^1 satisfy the growth conditions required in that theorem and the admissible control set $\mathcal{A}_0$ is not empty. More precisely, according to Theorem 5.15 in [3] we have that in the auxiliary problem there exists an optimal control in the class $\mathcal{A}_0$ and it is furthermore of the Markov (feedback) type, namely, as in (3.50). The existence of a Lipschitz Markov control can be obtained by using, e.g., results in [7] (see also [6]).

Before coming to the main result of the second part of this section, for later convenience we state the following lemma, whose proof can be obtained via a truncation argument analogous to that used in the proof of Theorem 2.1 concerning the uniqueness of the solution of (1.9).

LEMMA 3.9. *Any Lipschitz Markov control* $u_t = u_t(X_t, v_t)$ *belongs to the class* $\mathcal{L}_1$*; in particular, as a function of* y_0^t *it is Lipschitz in the sense of* (1.5).

THEOREM 3.10. *The following equality holds between the optimal values of the original and the auxiliary control problems*

$$\inf_{u \in \mathcal{L}_0} J(u) = \inf_{u \in \mathcal{L}_1} J'(u) = \inf_{\alpha \in \mathcal{A}_0^L} J(\alpha) \tag{3.53}$$

with $\mathcal{A}_0^L$ *as defined in* (3.37).

Furthermore, given an ε*-optimal control* $\alpha^\varepsilon \in \mathcal{A}_0^L$*, let* k *be so large that for the control* $u^{k,\varepsilon}$*, obtained via* (3.24) *from a Lipschitz control* $\alpha^{k,\varepsilon} \in \mathcal{A}^L$ *that in turn is obtained from* α^ε *via* (3.38)*, we have*

$$J'(u^{k,\varepsilon}) \leq J(\alpha^\varepsilon) + \varepsilon. \tag{3.54}$$

Then $u^{k,\varepsilon}$ *belongs to* $\mathcal{L}_1$ *and is a* 4ε*-optimal Lipschitz Markov control for the original problem.*

Proof. The first equality in (3.53) follows from Theorem 2.1. For the second equality note first that, letting u^ε be an ε-optimal control in $\mathcal{L}_1$, α^ε be the corresponding control in $\mathcal{A}_0$ obtained according to (3.11) and (3.6), and α^0 and $\alpha^{0,\varepsilon}$ be the optimal and nearly optimal controls of Theorem 3.8 and using Proposition 3.4 we obtain

$$\begin{aligned} \inf_{u \in \mathcal{L}_1} J'(u) &\geq J'(u^\varepsilon) - \varepsilon = J(\alpha^\varepsilon) - \varepsilon \geq \inf_{\alpha \in \mathcal{A}_0} J(\alpha) - \varepsilon \\ &= J(\alpha^0) - \varepsilon \geq J(\alpha^{0,\varepsilon}) - 2\varepsilon \geq \inf_{\alpha \in \mathcal{A}_0^L} J(\alpha) - 2\varepsilon. \end{aligned} \tag{3.55}$$

On the other hand let $\alpha^\varepsilon \in \mathcal{A}_0^L$ be ε-optimal. Starting from this α^ε, construct the Lipschitz control $\alpha^{k,\varepsilon} \in \mathcal{A}^L$ according to (3.38), and let $u^{k,\varepsilon}$ be the corresponding Lipschitz Markov control obtained according to (3.24). Then, using Corollary 3.7 and the fact that (see Lemma 3.9) if $u_t^{k,\varepsilon} = u_t^{k,\varepsilon}(X_t, v_t)$ is Lipschitz as a function of (X_t, v_t), then it is also in $\mathcal{L}_1$, we have for k sufficiently large

$$\inf_{\alpha \in \mathcal{A}_0^L} J(\alpha) \geq J(\alpha^\varepsilon) - \varepsilon \geq J'(u^{k,\varepsilon}) - 2\varepsilon \geq \inf_{u \in \mathcal{L}_1} J'(u) - 2\varepsilon. \tag{3.56}$$

Combining (3.55) with (3.56) one obtains

$$\inf_{u \in \mathcal{L}_1} J'(u) + 2\varepsilon \geq \inf_{\alpha \in \mathcal{A}_0^L} J(\alpha) \geq \inf_{u \in \mathcal{L}_1} J'(u) - 2\varepsilon, \tag{3.57}$$

from which, due to the arbitrariness of $\varepsilon > 0$, the second equality in (3.53) follows. From (3.55) and (3.56) one also obtains

$$J'(u^{k,\varepsilon}) \leq \inf_{u \in \mathcal{L}_1} J'(u) + 4\varepsilon, \tag{3.58}$$

i.e., the 4ε-optimality of $u^{k,\varepsilon}$.

Concluding remarks. From Theorem 3.10 we have that the optimal value $\inf_{u \in \mathcal{L}_0} J(u)$ of the original control problem can be determined by solving the auxiliary problem. However, as mentioned in Remark 3.5, while the auxiliary problem admits an optimal control, there may not exists a control for the original problem for which the optimal value is achieved. There are essentially two reasons for this:

- The filter process X_t corresponding to the optimal solution of the auxiliary problem may jump, so it can be represented as solution of (1.12) only if we allow the control also to have infinite power.
- Even if we allow infinite control power, the representation of the possible jumps of the filter process by means of (1.22) may require anticipative impulse controls.

In fact, due to the linearity in the control α of the Hamilton–Jacobi–Bellman equation of the auxiliary control problem, there will be intervals on which the optimal control α_s^0 for this latter problem will be either zero or one.

If $\alpha_s^0 = 1$, for the corresponding control u_t^0 of the original problem, obtained from α^0 via (3.24), we have $u_t^0 = 0$, and this motivated the extended study of the filter problem in section 1.2.

If, however, $\alpha_s^0 = 0$ on some interval $[s_1, s_2]$, the corresponding ν_s^0 (see (3.10c)) is constant, implying a jump for the inverse function $\Gamma_t^0 = \inf\{s : \nu_s^0 > t\}$. Since (see Lemma 3.2) $X_t = Z_{\Gamma_t}$, $v_t = \mu_{\Gamma_t}$, this then implies that X_t and v_t also jump and can therefore be a solution of (1.12) only if we allow controls u_t with infinite power. This implies an impulse control for the original problem at the moment $t_1 = \nu_{s_1}$, leading (see (1.1b)) to a discrete observation with intensity (see the relation $v_t = \mu_{\Gamma_t}$ in (3.23)) $\Delta v_{t_1} = s_2 - s_1$. Since s_2 is $\mathcal{F}_{s_2}^{Z,\nu} = \mathcal{F}_{t_1^+}^{X,v}$-measurable, this Δv_{t_1} cannot be determined on the basis of the observations up to time t_1; i.e., the control would be anticipative.

Our approach, based on the search of a nearly optimal control, avoids this problem. In fact, the control $u^{k,\varepsilon} \in \mathcal{L}_1$ obtained according to Theorem 3.10 (namely, obtained via (3.24) from a nearly optimal control $\alpha^{k,\varepsilon}$ of the auxiliary problem that belongs to $\mathcal{A}^L$ and thus satisfies $\alpha^{k,\varepsilon} > 0$) has finite power, is $\mathcal{F}_t^{X,v}$-measurable, and allows us to approximate arbitrarily closely the optimal value $\inf_{u \in \mathcal{L}_0} J(u) = \inf_{u \in \mathcal{L}_1} J'(u)$ of the objective function of the original problem.

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