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Binary extended perfect codes of length 16 of rank 14[1]

Abstract

We continue the classification of binary extended nonlinear perfect codes of length 16. In the previous papers we enumerated all such codes of ranks at most thirteen over the field $\mathbb{F}_2$. Here we classified the extended binary perfect $(16, 4, 2^{11})$-codes of rank 14 over $\mathbb{F}_2$. We proved that among non-equivalent extended binary perfect $(16, 4, 2^{11})$-codes there are exactly 1708 non-equivalent codes with rank 14 over $\mathbb{F}_2$. Among these codes there are 844 codes, classified by Phelps (Solovieva-Phelps codes) and 864 other codes obtained by construction of Etzion-Vardy and by generalized doubling construction. Thus, the only open question in classification of extended binary perfect $(16, 4, 2^{11})$-codes is classification of such codes of rank 15 over $\mathbb{F}_2$.

§ 1. Introduction

One of the interesting open problems of algebraic coding theory is *the classification of nonlinear binary perfect codes with Hamming parameters*. Even for the smallest nontrivial length $n = 15$ or $n = 16$ (for the extended codes) this problem is very far from the full solution. There are several papers dedicated to the characterization of such codes constructed by several methods. The first family of the binary perfect nonlinear $(n = 2^m - 1, d = 3, N = 2^{n-m})$ codes was described by Vasiliev in [1]. Hergert [2] has found all non-equivalent Vasiliev's codes of length 15: there are 19 such non-equivalent codes (including the linear code). Malugin [3] showed that the number of non-equivalent extended Vasiliev's codes of length 16 is equal to 13. He classified also in [3] all the nonlinear perfect codes of length

[1]Supported in part by the Russian Foundation for Basic Research, project no. 03 - 01 - 00098

16, which can be obtained from the Hamming code by the simultaneous translate of non-overlapping components of different directions. The number of non-equivalent such codes of length 15 occurs to be equal to 370, and the number of such extended perfect codes of length 16 is equal to 248. Solov'eva [4] and Phelps [5] derived the construction, which doubles the length of code (in common terminology, symmetric X4 construction). Phelps [6] proved that there are exactly 963 non-equivalent binary extended perfect codes (including the linear code) of length 16, obtained by this construction. Etzion and Vardi [7] showed that there are codes of rank 14, which can not be obtained by the Solovieva - Phelps doubling construction. They generalized this construction and gave examples of such codes of rank 14, which can not be obtained by the Solovieva-Phelps doubling construction.

This paper is a natural continuation of our previous results [8, 9] where we started the systematic investigation of codes of length 15 and 16 with given rank over the field $\mathbb{F}_2$. In [8] we have described all non-equivalent binary extended perfect nonlinear codes $(16, 4, 2^{11})$ of rank at most 13 over $\mathbb{F}_2$. All such codes can be obtained by the generalized concatenated (GC) construction [10-13]. Our results of paper [8] can be formulated as follows. *Among non-equivalent binary extended perfect codes of length* $n = 16$ *there are exactly*:

– *one code of rank* 11 *(the Hamming code)*;

– 12 *codes of rank* 12 *(the Vasiliev codes)*;

– 272 *codes of rank* 13 *(the GC-codes with the length of inner codes* $n_b = 4$ *only).*

In paper [9] we have classified all non-equivalent binary perfect codes of length $n = 15$. In particular, we have proved that *among such codes there are:*

– *one code of rank* 11 *(the Hamming code)*;

– 18 *codes of rank* 12 *(the Vasiliev codes)*;

– 758 *codes of rank* 13 *(the GC-codes with the length of inner codes* $n_b = 4$ *only).*

The purpose of this paper is to classify all non-equivalent binary extended perfect codes of length 16 of rank 14 over $\mathbb{F}_2$. We describe the general doubling construction of extended binary perfect codes of length $n = 16$ and of rank 14 over $\mathbb{F}_2$, which generalize the construction of Etzion and Vardi [7]. This new construction produces all such codes of length 16 with a rank 14 over $\mathbb{F}_2$. We can formulate the result of our paper as follows. *Among the binary extended perfect codes of length* $n = 16$ *there are exactly:*

– *one code of rank* 11 *(the Hamming code);*

– 12 *codes of rank* 12 *(the Vasiliev codes);*

– 272 *codes of rank* 13 *(the GC-codes with the length of inner codes* $n_b = 4$ *only);*

– 1708 *codes of rank* 14*, in particular,* 844 *Solovieva-Phelps codes and* 864 *codes, obtained by Etzion-Vardi construction and its generalization.*

The number of such codes of rank 15 is still remain open.

The paper is organized as follows. In § 2 we give some notations and terminology. In § 3 we describe the Solovieva-Phelps construction (symmetric X4 construction) and the doubling construction of Etzion-Vardi of extended binary perfect $(n, 4, 2^{n-m-1})$-codes of length $n = 2^m$. In § 4 we give some preliminary results concerning the structure of extended binary perfect $(16, 4, 2^{11})$-codes with rank 14 over $\mathbb{F}_2$. The general doubling construction of codes of length 16 and rank 14 is given in § 5. The paragraph § 6 is dedicated to construction of all canonical $(16, 4, 2^{11})$-codes with rank 14 over $\mathbb{F}_2$. All together there are exactly 10312 such distinct codes with kernels of sizes from 4 to 256. We give the number of such codes for each size of the kernel. In § 7 we give the main results: classification of the non-equivalent extended binary perfect codes of length 16 with rank 14 over $\mathbb{F}_2$.

§ 2. Preliminary results and terminology

Let E be a finite alphabet of size q : $E = \{0, 1, ..., q-1\}$. A q-ary code of length n is an arbitrary subset of E^n. Denote such q-ary code C with length n, with the minimal distance d and cardinality N as $(n, d, N)_q$-code and as (n, d, N)-code we denote such binary code C, i.e. when $q = 2$. Denote by $\mathrm{wt}(\boldsymbol{x})$ the Hamming weight of vector $\boldsymbol{x}$ over E. For a binary (i.e. $q = 2$) code C denote by $\langle C \rangle$ the linear envelope of words of C over $\mathbb{F}_2$. The dimension of space $\langle C \rangle$ is called the *rank* of C over $\mathbb{F}_2$ and is denoted $\mathrm{rank}(C)$. For a binary code C call a *kernel* and denote $\ker(C)$ the set of all vector $\boldsymbol{x}$ from E^n stabilizing this code: $C + \boldsymbol{x} = \{\boldsymbol{c} + \boldsymbol{x} : \ \boldsymbol{c} \in C\} = C$ where + denotes component wise addition modulo two. It is clear, that $\ker(C)$ is a linear space and for the code C with zero vector $\ker(C)$ is a subset of C.

For any two subsets Y and Z of E^n denote by $d(Y,Z)$ the minimal distance between Y and Z:

$$d(Y,Z) \;=\; \min\{d(\boldsymbol{y},\boldsymbol{z}):\; \boldsymbol{y}\in Y,\; \boldsymbol{z}\in Z\}.$$

For vector $\boldsymbol{v}=(v_1,...,v_n)\in E^n$ denote by $\mathrm{supp}(\boldsymbol{v})$ its support, i.e. the set of indices with nonzero positions:

$$\mathrm{supp}(\boldsymbol{v}) = \{i:\; v_i\neq 0\}.$$

Denote by $\bar{\boldsymbol{v}}$ a vector, which is a complementary to $\boldsymbol{v}$, i.e. $\bar{v}_i = v_i+1$.

If $E=\mathbb{F}_q$ is a finite field of order q, the q-ary (n,d,N)-code A which is a linear k-dimensional space over $\mathbb{F}_q$ is denoted by $[n,k,d]_q$-code and by $[n,k,d]$ code for $q=2$. For binary vectors $\boldsymbol{x}=(x_1,\cdots,x_n)$ and $\boldsymbol{y}=(y_1,\cdots,y_n)$ denote by $(\boldsymbol{x}\cdot\boldsymbol{y}) = x_1y_1+\cdots+x_ny_n$ their inner product over $\mathbb{F}_2$. For a linear $[n,k,d]$-code A denote by $A^\perp$ its dual code:

$$A^\perp \;=\; \{\boldsymbol{v}\in\mathbb{F}_2^n:\; (\boldsymbol{v}\cdot\boldsymbol{c})=0,\; \forall\,\boldsymbol{c}\in A\}.$$

It is clear that $A^\perp$ is a linear $[n,n-k,d^\perp]$ code with some minimal distance $d^\perp$.

Denote by E^n_{ev} the set of all binary vectors of length n of even weight. Let $J_n=\{1,2,...,n\}$ be the coordinate set of E^n and let S_n be the full group of permutations of n elements. For any $i\in J_n$ and $\pi\in S_n$, define the image of i under the action of π by $\pi(i)$. For any set X of E^n and any $\pi\in S_n$ denote $\pi X=\{\pi(\boldsymbol{x}):\; \boldsymbol{x}\in X\}$.

Define the action of E^n_{ev} on itself by shifts, i.e. for any $\boldsymbol{y}\in E^n_{\mathrm{ev}}$ and $\boldsymbol{h}\in E^n_{\mathrm{ev}}$, set $\boldsymbol{h}(\boldsymbol{y})=\boldsymbol{h}+\boldsymbol{y}$. We denote this group of actions by H^n_{ev}. Note that shifts are equivalent to coordinate-wise permutations of the binary alphabet.

Let $G_n = <S_n, H^n_{\mathrm{ev}}>$ be the group generated by H^n_{ev} and S_n. Then $G=S_n\rtimes H^n_{\mathrm{ev}}$ is the semi-direct product of S_n and H^n_{ev}, where H^n_{ev} is a normal subgroup of G_n.

§ 3. Constructions of Solovieva-Phelps and Etzion-Vardi

For extended binary perfect codes Solovieva-Phelps construction (or, symmetric X4-construction) looks as follows. Let $n=2^m$ and let E^n_{ev} be the even subspace of E^n. Assume that $E^n_{\rm ev}$ can be presented in two ways as follows:

$$E^n_{ev} \;=\; \bigcup_{i=1}^{n} A_i \;=\; \bigcup_{j=1}^{n} B_j,$$

where A_i and B_j are extended binary perfect codes of length n for all $i = 1, ..., n$ and $j = 1, ..., n$. Then the following statement is valid [4, 5].

Proposition 1 (*Solovieva* [4] - *Phelps* [5] *construction*). *Let $\pi \in S_n$ be any permutation. Then the set*

$$X \;=\; \bigcup_{i=1}^{n} \{(\boldsymbol{a}\,|\,\boldsymbol{b}) :\; \boldsymbol{a} \in A_i,\; \boldsymbol{b} \in B_{\pi(i)}\}$$

is an extended binary perfect $(2n, 4, 2^{2n-m-2})$*-code.*

This construction occurs to be very useful for the construction of codes with the given rank and kernel (see [13]). In particular, it is possible to construct the codes of length $n = 2^m$ with rank $n-2$ over $\mathbb{F}_2$.

In [6] Phelps enumerated a large class of the extended binary perfect $(16, 4, 2^{11})$-codes, obtained by Solovieva-Phelps construction. In order to formulate his results we need some definitions. Let $A_1, ..., A_n$ and $B_1, ..., B_n$ be two partitions of $E^n_{\rm ev}$ into extended perfect $(n, 4, 2^{n-1-m})$-codes A_i and B_j where $n = 2^m$. Two such partitions $A_1, ..., A_n$ and $B_1, ..., B_n$ of $E^n_{\rm ev}$ are equivalent, if there is an element $\boldsymbol{g} \in G_n$ and a permutation $\tau \in S_n$ such that

$$(\boldsymbol{g}A_1, ..., \boldsymbol{g}A_n) \;=\; (B_{\tau^{-1}(1)}, ..., B_{\tau^{-1}(n)}).$$

Phelps [6] proved that there are exactly 10 non-equivalent partitions of $E^8_{\rm ev}$, which we denoted here by L_0, L_1, ..., L_9, connected with Phelps partitions P_i as follows:

$$L_i \;=\; P_i,\; i = 0, 1, 2, 3, 4, 5, 6, \;\text{ and }\; L_i \;=\; P_{i+1},\; i = 7, 8, 9. \tag{1}$$

We say that L_0, L_1, ..., L_9 are *canonical partitions.*

Definition 1 *Denote by* $\mathrm{Stab}_{G_8}(L_k)$ *the stabilizer of* L_k *in* G_8 *and by* Q_k *a group of permutations of its components induced by automorphisms from* G_8 (*found by Phelps* [6] *for all non-equivalent* (*canonical*) *partitions* L_k, $k = 0, 1, \ldots, 9$) :

$$Q_k \;=\; \{\pi \in S_8 :\; \exists \boldsymbol{g} \in \mathrm{Stab}_{G_8}(L_k) :\; \boldsymbol{g}L_{k,s} = L_{k,\pi^{-1}(s)},\; k = 0, 1, \ldots, 7\}.$$

Using all non-equivalent partitions $L_0, \ldots, L_9$ of E^8_{ev}, Phelps [6] defined the following set of Solovieva-Phelps codes of length 16.

Definition 2 (*Solovieva-Phelps codes*). *Let L_i and L_j be any two partitions from the set $\{L_0, L_1, \ldots, L_9\}$ and let $\pi \in S_8$ be any permutation. Denote by $C = C(L_i, L_j, \pi)$ the resulting extended perfect $(16, 4, 2^{11})$-code, obtained by Proposition 1. Denote by $\mathcal{C}_{SP}$ the set of all codes $C(L_i, L_j, \pi)$ when L_i and L_j run over all set $\{L_0, L_1, \ldots, L_9\}$ and when π runs over S_8.*

Theorem 1 (*Phelps* [6]). *The set $\mathcal{C}_{SP}$ of codes $C(L_i, L_j, \pi)$ consists of exactly 963 non-equivalent extended perfect binary $(16, 4, 2^{11})$-codes, namely including:*

– 1 *code of rank* 11 (*the Hamming code*);

– 7 *codes of rank* 12 (*the Vasiliev codes*);

– 110 *codes of rank* 13 (*the GC-codes with length of the inner codes $n_b = 4$*);

– 845 *codes of rank* 14.

Classifying all codes of length 16 rank 14, we repeated all computations for the codes from the set $\mathcal{C}_{SP}$ of rank 14 over $\mathbb{F}_2$, and we have found one omission in the results of Phelps [6].

Theorem 2 *The set $\mathcal{C}_{SP}$ of codes $C(L_i, L_j, \pi)$ consists of exactly 844 non-equivalent extended perfect binary $(16, 4, 2^{11})$-codes of rank 14 over $\mathbb{F}_2$.*

Proof. From the table 2 of paper [6] we have that the number of codes of the type $C(L_7, L_7, \pi)$ is equal to 5 (in notation of [6], it is codes, formed by partitions $P_i = P_j = 8$). These codes have the numbers 10600, 10601, 10700, 10900, and 11500, and defined by the following permutations π_i $(i = 1, l \ldots, 5)$:

$$(01235467), \quad (67012354), \quad (01234567), \quad (06712345), \quad (67123450)$$

respectively. The group Q_8 is induced by the permutations [6]

$$(1, 2, 6)(3, 5, 7), \quad (0, 1, 3)(2, 7, 5), \quad (0, 1)(2, 5)(3, 4)(6, 7).$$

It is easy to see that the two first codes 10600, 10601, corresponding to the permutations π_1 and π_2 are equivalent to each other, since they belong to the same double (Q_8-Q_8)-coset of S_8. We add here that all these codes have the kernel 4. $\triangle$

Etzion and Vardi [7] generalized the Solovieva-Phelps construction in the following way. Let V be a subset of E^n_{ev} where $n = 2^m$. Let $\mathcal{A} = (A_1, A_2, ..., A_k)$ and $\mathcal{B} = (B_1, B_2, ..., B_k)$ be two ordered sets of subsets of V. For $\boldsymbol{v} \in V$, define

$$\Lambda_A(\boldsymbol{v}) = \{i : \ \boldsymbol{v} \in A_i\}, \quad \Lambda_B(\boldsymbol{v}) = \{i : \ \boldsymbol{v} \in B_i\},$$

where $A_i \in \mathcal{A}$ and $B_i \in \mathcal{B}$. We say that $\mathcal{A}$ and $\mathcal{B}$ form a *perfect segmentation* of order k of the set V, if the following both sets

$$\bigcup_{i \in \Lambda_B(\boldsymbol{v})} A_i \quad \text{and} \quad \bigcup_{i \in \Lambda_A(\boldsymbol{v})} B_i$$

are extended perfect codes of length n for all $\boldsymbol{v} \in V$.

Proposition 2 (*Etzion−Vardi* [7] *construction*). *Let $n = 2^m$ and let $\mathcal{A}$ and $\mathcal{B}$ be a perfect segmentation of E^n_{ev} of order n. Then the set*

$$C \ = \ \bigcup_{i=1}^{n} \{(\boldsymbol{a} \,|\, \boldsymbol{b}) : \ \boldsymbol{a} \in A_i, \ \boldsymbol{b} \in B_i\}$$

is an extended perfect code of length $2n$.

In [7] Etzion and Vardi proved that the construction above provides extended perfect codes of length 16 and rank 14, which are not equivalent to any codes, obtained by Solovieva-Phelps construction, given by proposition 1 above and also to the extended perfect codes, obtained in [14].

§ 4. Basic properties of $(16, 4, 2^{11})$ codes of rank 14

Let C be an arbitrary extended binary perfect $(16, 4, 2^{11})$-code of rank 14 over $\mathbb{F}_2$. We consider the general properties of such code.

Applying the appropriate permutation of coordinates, C can be presented in the form, when the $[16, 8, 2]$-code $C^\perp$, dual to C, looks as follows:

$$C^\perp \;=\; \{\boldsymbol{u}_0, \boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_1 + \boldsymbol{u}_2\}, \tag{2}$$

where $\boldsymbol{u}_0$ is the zero vector, $\boldsymbol{u}_1 = (1111111100000000)$, and $\boldsymbol{u}_2 = (0000000011111111)$. Thus we split coordinates into two blocks of eight coordinates such that any $\boldsymbol{c} \in C$ consists of two vectors $\boldsymbol{c} = (\boldsymbol{c}_1 \,|\, \boldsymbol{c}_2)$ where each vector $\boldsymbol{c}_i$ satisfies to the overall parity checking:

$$\mathrm{wt}(\boldsymbol{c}_i) \;\equiv 0 \pmod 2, \quad i = 1, 2$$

(we call it a *parity rule*). The group (subgroup of S_{16}) of two elements which permutes the blocks is identified with S_2. An element $\boldsymbol{g}_1 \times \boldsymbol{g}_2 \in G_8 \times G_8 \subset G_{16}$ acts on $(\boldsymbol{x} \,|\, \boldsymbol{y})$ in the natural way:

$$(\boldsymbol{g}_1 \times \boldsymbol{g}_2)(\boldsymbol{x} \,|\, \boldsymbol{y}) \;=\; (\boldsymbol{g}_1(\boldsymbol{x}) \,|\, \boldsymbol{g}_2(\boldsymbol{y})).$$

Definition 3 *Define the group:*

$$\begin{aligned} G \;&=\; S_2 \rtimes (G_8 \times G_8) \\ &=\; S_2 \rtimes ((S_8 \rtimes H^8_{ev}) \times (S_8 \rtimes H^8_{ev})) \\ &=\; (S_2 \rtimes (S_8 \times S_8)) \rtimes (H^8_{ev} \times H^8_{ev}). \end{aligned}$$

It is clear that G is a subgroup of G_{16}. Then we have the following statement.

Lemma 1 *Let C be an arbitrary extended binary perfect $(16, 4, 2^{11})$-code of rank 14 over $\mathbb{F}_2$ with dual code (2). Suppose there exists a permutation $\sigma \in S_{16}$ so that σC satisfies the parity rule. Then $\sigma \in S_2 \rtimes (S_8 \times S_8)$.*

Proof. Since C satisfies parity rule, we have that

$$(\boldsymbol{x} \cdot \boldsymbol{u}_1) \;=\; 0, \tag{3}$$

for any $\boldsymbol{x} \in C$. Similarly, since σC satisfies the parity rule, we have that

$$(\sigma(\boldsymbol{x}) \cdot \boldsymbol{u}_1) = 0, \quad \text{for any } \boldsymbol{x} \in C.$$

Multiplying both vectors $\sigma(\boldsymbol{x})$ and $\boldsymbol{u}_1$ by σ^{-1}, we obtain

$$(\boldsymbol{x} \cdot \sigma^{-1}(\boldsymbol{u}_1)) = 0, \quad \text{for any } \boldsymbol{x} \in C. \tag{4}$$

Let $\boldsymbol{u}' = \boldsymbol{u}_1 + \sigma^{-1}(\boldsymbol{u}_1)$. From (3) and (4) we have that

$$(\boldsymbol{x} \cdot \boldsymbol{u}') = 0, \quad \text{for any } \boldsymbol{x} \in C.$$

Thus $\boldsymbol{u}' \in C^\perp$ and consequently (recall that $C^\perp$ is a vector space) $\sigma^{-1}(\boldsymbol{u}_1) \in C^\perp$. Taking into account that $\sigma^{-1}(\boldsymbol{u}_1)$ is of weight 8, we obtain that $\sigma^{-1}(\boldsymbol{u}_1)$ is equal to either $\boldsymbol{u}_1$ or $\boldsymbol{u}_2$. So $\sigma(\boldsymbol{u}_1) = \boldsymbol{u}_1$ or $\sigma(\boldsymbol{u}_2) = \boldsymbol{u}_1$, in other words, σ either stabilizes the blocks or permutes them. $\triangle$

Recall that E^8_{ev} is the subspace of E^8, formed by the vectors of even weight. Denote any codeword of C by $\boldsymbol{c} = (\boldsymbol{a} \,|\, \boldsymbol{b})$.

Definition 4 *Let C be a $(16, 4, 2^{11})$ code of rank 14 over $\mathbb{F}_2$ with dual code (2). Denote by $A_\ell(\boldsymbol{a})$ (respectively, by $A_r(\boldsymbol{b})$) the sets obtained by fixing vector $\boldsymbol{a}$ (respectively $\boldsymbol{b}$) :*

$$A_r(\boldsymbol{a}) \;=\; \{\boldsymbol{b} : \; (\boldsymbol{a} \,|\, \boldsymbol{b}) \in C\}, \quad A_\ell(\boldsymbol{b}) \;=\; \{\boldsymbol{a} : \; (\boldsymbol{a} \,|\, \boldsymbol{b}) \in C\}.$$

Lemma 2 *Suppose the conditions of lemma 1 are satisfied. Let $\boldsymbol{c} = (\boldsymbol{a} \,|\, \boldsymbol{b})$ be any codeword of C. Then the set $A_\ell(\boldsymbol{b})$ (respectively $A_r(\boldsymbol{a})$) is an extended binary perfect $(8, 4, 16)$-code.*

Proof. The fact that $A_\ell(\boldsymbol{b})$ (respectively, $A_r(\boldsymbol{a})$) has the minimal distance 4 follows from definition of such set. The cardinality follows from counting arguments. In average, over all $\boldsymbol{b} \in E^8_{\mathrm{ev}}$, we have that

$$|\bar{A}_\ell| \;=\; \frac{1}{|E^8_{\mathrm{ev}}|} \times \sum_{\boldsymbol{b} \in E^8_{\mathrm{ev}}} |A_\ell(\boldsymbol{b})| \;=\; \frac{|C|}{|E^8_{\mathrm{ev}}|} \;=\; 16.$$

From the other side, $|A_\ell(\boldsymbol{b})|$ can not be more than 16 for any $\boldsymbol{b} \in E^8_{\mathrm{ev}}$. Thus $|A_\ell(\boldsymbol{b})| = 16$. Similarly, the same equality is valid for $|A_r(\boldsymbol{a})|$. $\triangle$

Definition 5 *Define the extended ball $W(\boldsymbol{x})$ of radius two centered at $\boldsymbol{x} \in E^8_{ev}$:*

$$W(\boldsymbol{x}) \;=\; \boldsymbol{x} \;+\; W_0 \;=\; \{\boldsymbol{x} + \boldsymbol{w} : \; \boldsymbol{w} \in W_0\}$$

where $W_0 = \{\boldsymbol{e}_0, \boldsymbol{e}_1, ..., \boldsymbol{e}_7\}$ *and*

$$\begin{array}{rclrcl} \boldsymbol{e}_0 & = & (00000000), & \boldsymbol{e}_1 & = & (00000011), \\ \boldsymbol{e}_2 & = & (00000101), & \boldsymbol{e}_3 & = & (00001001), \\ \boldsymbol{e}_4 & = & (00010001), & \boldsymbol{e}_5 & = & (00100001), \\ \boldsymbol{e}_6 & = & (01000001), & \boldsymbol{e}_7 & = & (10000001), \end{array}$$

in this notation $W_0 = W(\boldsymbol{e}_0)$. Note that the stabilizer of W_0 in S_8 fixes the last coordinate and is isomorphic to S_7. In this paper, we identify S_7 with $\mathrm{Stab}_{S_8}(W_0)$ (i.e. S_7 permutes the first seven coordinates).

Lemma 3 *Suppose we are in conditions of lemma 1 and let* $(\boldsymbol{a}_1 \,|\, \boldsymbol{b}_1)$ *and* $(\boldsymbol{a}_2 \,|\, \boldsymbol{b}_2)$ *be any two codewords of* C*. Let* $\boldsymbol{a}_1$ *and* $\boldsymbol{a}_2$ *(respectively,* $\boldsymbol{b}_1$ *and* $\boldsymbol{b}_2$*) be such that* $d(\boldsymbol{a}_1, \boldsymbol{a}_2) = 2$ *(respectively,* $d(\boldsymbol{b}_1, \boldsymbol{b}_2) = 2$*). Then the corresponding codes* $A_r(\boldsymbol{a}_1)$ *and* $A_r(\boldsymbol{a}_2)$ *(respectively,* $A_\ell(\boldsymbol{b}_1)$ *and* $A_\ell(\boldsymbol{b}_2)$*) do not intersect each other, i.e.* $A_r(\boldsymbol{a}_1) \cap A_r(\boldsymbol{a}_2) = \emptyset$ *(respectively,* $A_\ell(\boldsymbol{b}_1) \cap A_\ell(\boldsymbol{b}_2) = \emptyset$*).*

Proof. In contrary, assume that there is $\boldsymbol{x}$ such that $\boldsymbol{x} \in A_r(\boldsymbol{a}_1) \cap A_r(\boldsymbol{a}_2)$. Then we have

$$d((\boldsymbol{a}_1 \,|\, \boldsymbol{x}), (\boldsymbol{a}_2 \,|\, \boldsymbol{x})) \;=\; d(\boldsymbol{a}_1, \boldsymbol{a}_2) \;=\; 2,$$

i.e. a contradiction, since $(\boldsymbol{a}_1 \,|\, \boldsymbol{x})$ and $(\boldsymbol{a}_2 \,|\, \boldsymbol{x})$ are distinct codewords of C. The proof of the second statement is similar. △

Lemma 4 *Suppose we are in conditions of lemma 1 and let* $\boldsymbol{x} \in E^8_{ev}$ *be any vector. Consider the codes* $A_\ell(\boldsymbol{x}_i)$ *where* $\boldsymbol{x}_i \in W(\boldsymbol{x})$*,* $i = 0, 1, ..., 7$ *and* $\boldsymbol{x}_0 = \boldsymbol{x}$*. Then the set of codes* $A_\ell(\boldsymbol{x}_0)$*,* $A_\ell(\boldsymbol{x}_1)$*, ...,*$A_\ell(\boldsymbol{x}_7)$ *is a partition of* E^8_{ev}*.*

Proof. Since

$$|W(\boldsymbol{x})| \cdot |A_\ell(\boldsymbol{x}_i)| \;=\; |E^8_{\mathrm{ev}}| \;=\; 2^7,$$

we have to check only that any two distinct codes $A_\ell(\boldsymbol{x}_i)$ and $A_\ell(\boldsymbol{x}_j)$ where $i \neq j$ and $i, j \in \{0, 1, ..., 7\}$ have empty intersection. But this follows from lemma 3, since for any $\boldsymbol{x}_i, \boldsymbol{x}_j$ from $W(\boldsymbol{x})$ we have that $d(\boldsymbol{x}_i, \boldsymbol{x}_j) = 2$. △

Lemma 5 *Suppose we are in conditions of lemma 1 and let* $\boldsymbol{a}$ *and* $\bar{\boldsymbol{a}}$ *(respectively* $\boldsymbol{b}$ *and* $\bar{\boldsymbol{b}}$*) be any pair of complementary vectors from* E^8_{ev}*. Let* $A_r(\boldsymbol{a})$ *and* $A_r(\bar{\boldsymbol{a}})$ *(respectively* $A_\ell(\boldsymbol{b})$ *and* $A_\ell(\bar{\boldsymbol{b}})$*) be the corresponding codes. Then these two codes are coincide:* $A_r(\boldsymbol{a}) = A_r(\bar{\boldsymbol{a}})$ *(respectively* $A_\ell(\boldsymbol{b}) = A_\ell(\bar{\boldsymbol{b}})$*).*

Proof. In contrary, assume that it is not the case, i.e. there is $\boldsymbol{x} \in E^8_{\mathrm{ev}}$ such that $A_r(\boldsymbol{x}) \neq A_r(\bar{\boldsymbol{x}})$. Since $A_r(\boldsymbol{x})$ is an extended perfect code with minimal distance 4, this means that there are two vectors $\boldsymbol{y} \in A_r(\boldsymbol{x})$ and $\boldsymbol{z} \in A_r(\bar{\boldsymbol{x}})$ such that $d(\boldsymbol{y}, \boldsymbol{z}) = 2$. Since $A_r(\bar{\boldsymbol{x}})$ is also an extended perfect code, it contains vector $\bar{\boldsymbol{z}}$ such that $d(\boldsymbol{y}, \bar{\boldsymbol{z}}) = 6$. Consider now the following two vectors from C: $(\boldsymbol{x} \,|\, \boldsymbol{y})$ and $(\bar{\boldsymbol{x}} \,|\, \bar{\boldsymbol{z}})$. We have that

$$d((\boldsymbol{x} \,|\, \boldsymbol{y}), (\bar{\boldsymbol{x}} \,|\, \bar{\boldsymbol{z}})) \;=\; d(\boldsymbol{x}, \bar{\boldsymbol{x}}) + d(\boldsymbol{y}, \bar{\boldsymbol{z}}) \;=\; 8 + 6 \;=\; 14.$$

But C is an extended binary perfect code too, and, therefore, for any $\boldsymbol{c} \in C$ there is a complementary word $\bar{\boldsymbol{c}} \in C$. Thus, we obtain two vectors $(\bar{\boldsymbol{x}}|\bar{\boldsymbol{y}})$ and $(\bar{\boldsymbol{x}}|\bar{\boldsymbol{z}})$ at distance 2 from each other, i.e. a contradiction. The proof of the second statement is similar. $\triangle$

Remark 1 *It is easy to see that the results above, which we derived for* $(16, 4, 2^{11})$*-codes of rank* 14 *over* $\mathbb{F}_2$*, are valid for any* $(n, 4, 2^{n-m-1})$*-code* C *of arbitrary length* $n = 2^m \geq 16$ *with rank* $n - 2$ *over* $\mathbb{F}_2$*.*

§ 5. Construction of extended $(16, 4, 2^{11})$-codes of rank 14

Now we describe the construction of the extended binary perfect $(16, 4, 2^{11})$-codes C of rank 14. Let $\mathcal{C}$ be the set of all such distinct codes C. Our purpose is to parameterize all these codes, using the canonical partitions. We can do it using the special subsets of code C, called headings, formed by the two partitions, connected with the two spheres W_0 which occur on the left and right hand sides (the first and the second blocks) of the codewords. We start with the definition of *heading* of a code. Clearly when $\boldsymbol{c} = (\boldsymbol{a} \,|\, \boldsymbol{b})$ runs over C, each of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ run over the set $E^8_{\rm ev}$. In particular, when $\boldsymbol{a}$ runs over the ball W_0 the corresponding codes $A_r(\boldsymbol{a})$ form a partition of $E^8_{\rm ev}$,

$$E^8_{\rm ev} \;=\; \bigcup_{\boldsymbol{a}\in W_0} A_r(\boldsymbol{a}) \;=\; \bigcup_{i=0}^{7} A_r(\boldsymbol{e}_i).$$

Similarly, when $\boldsymbol{b}$ runs over the set W_0, the codes $A_\ell(\boldsymbol{b})$ also form a partition of $E^8_{\rm ev}$. Denote by Ω the set of all distinct partitions $L = (A_0, A_1, \ldots, A_7)$ of $E^8_{\rm ev}$ into extended binary perfect codes, i.e. A_s is a $(8, 4, 16)$-code Thus we have

$$\Omega \;=\; \bigcup_{i=0}^{9} \mathrm{Orb}_{G_8}(L_i).$$

Moreover the following result holds.

Proposition 3 (*Computational result*). *There exist exactly* 27330 *different partitions of* E^8_{ev} *which can be arranged under action of* G_8 *into ten orbits* $\mathrm{Orb}_{G_8}(L_i)$*, where* $i = 0, \ldots, 9$ *of sizes*

$$\{840, 420, 5040, 5040, 5040, 30, 1680, 1920, 630, 6720\}$$

ordered according to the indices i *of* $\mathrm{Orb}_{G_8}(L_i)$*.*

Definition 6 *Let* C *be a* $(16, 4, 2^{11})$ *code with rank* 14 *over* $\mathbb{F}_2$*. Define the following subset* $F = F(C)$ *of* C *(of* 248 *words), consisting of two partitions with* 8 *common words counted twice*

$$F(C) \;=\; \bigcup_{s=0}^{7}\{(\boldsymbol{e}_s\,|\,\boldsymbol{y}):\; \boldsymbol{y}\in A_r(\boldsymbol{e}_s)\} \;\bigcup\; \bigcup_{s=0}^{7}\{(\boldsymbol{x}\,|\,\boldsymbol{e}_s):\; \boldsymbol{x}\in A_l(\boldsymbol{e}_s)\}. \qquad (5)$$

We say that C *has a heading* F *and for the sake of simplicity write as:*

$$F \;=\; \bigcup_{s=0}^{7} \boldsymbol{e}_s\times A_r(\boldsymbol{e}_s) \;\bigcup\; \bigcup_{s=0}^{7} A_l(\boldsymbol{e}_s)\times \boldsymbol{e}_s.$$

Assume that the partition $A_\ell(\boldsymbol{e}_0)$, $A_\ell(\boldsymbol{e}_1)$, ..., $A_\ell(\boldsymbol{e}_7)$ is equivalent to L_i for some i, $i = 0, 1, ..., 9$ and the partition $A_r(\boldsymbol{e}_0)$, $A_r(\boldsymbol{e}_1)$, ..., $A_r(\boldsymbol{e}_7)$ is equivalent to L_j for some j, $j = 0, 1, ..., 9$. Recall that L_i and L_j are among of the ten canonical (non-equivalent) partitions of Phelps [6]. All these partitions L_k, $k = 0, 1, ..., 9$ are ordered, according to the vectors $\boldsymbol{e}_s$ of the ball W_0:

$$L_k = (L_{k,0}, L_{k,1}, \ldots, L_{k,7}) \text{ where } \boldsymbol{e}_s \in L_{k,s} \text{ for } s = 0, 1, \ldots, 7.$$

Without loss of generality we can assume that $i \leq j$ (if not we can consider the code C' obtained from C on switching the sides). Furthermore, by the corresponding shift and permutation of coordinates we can obtain the following ordering of L_i:

$$L_i = (L_{i,0}, L_{i,1}, ..., L_{i,7}), \quad L_{i,s} = A_\ell(\boldsymbol{e}_s), \tag{6}$$

where the vectors $\boldsymbol{e}_s$ $(s = 0, 1, ..., 7)$ are given by definition 5. In such way we arrive to the following natural *canonical heading*. For $\boldsymbol{a} \in E^n$ and $X \subseteq E^n$ denote:

$$\boldsymbol{a} \times X = \{(\boldsymbol{a} \,|\, \boldsymbol{x}) : \boldsymbol{x} \in X\}, \quad X \times \boldsymbol{a} = \{(\boldsymbol{x} \,|\, \boldsymbol{a}) : \boldsymbol{x} \in X\}.$$

Definition 7 (*Canonical* (i, j, k) *heading*). *Let* $0 \leq i \leq j \leq 9$ *and* L_i, L_j *are two canonical partitions. Define the set of* 248 (*where* 8 *words are counted twice*) *elements as follows*:

$$\begin{aligned} F_{i,j}^{(k)} &= \bigcup_{s=0}^{7} \boldsymbol{e}_{\pi_k^{-1}(s)} \times L_{j,s} \bigcup \bigcup_{s=0}^{7} L_{i,\pi_k^{-1}(s)} \times \boldsymbol{e}_s \\ &= \bigcup_{s=0}^{7} \{(\boldsymbol{e}_{\pi_k^{-1}(s)}|x) : x \in L_{j,s}\} \bigcup \bigcup_{s=0}^{7} \{(y \,|\, \boldsymbol{e}_s) : y \in L_{i,\pi_k^{-1}(s)}\}. \end{aligned}$$

where $k = 1, 2, ..., m(i, j)$, *and*

$$\{\pi_1, , \pi_2, ..., \pi_{m(i,j)}\}$$

is a fixed set of the $(Q_j\text{-}Q_i)$ *double-coset representatives of* S_8.

The next statement gives all canonical headings (i, j, k) as sizes $(Q_j\text{-}Q_i)$ double-cosets of the group S_8.

Proposition 4 (*Computational result*). *There exist* 1050 *different canonical headings* (i, j, k), *which can be arranged into the following* $(Q_j\text{-}Q_i)$ *double-cosets of* S_8 *with sizes* $m(i, j)$ *given* *by the following table*:

$j \setminus i$	0	1	2	3	4	5	6	7	8	9
0	16									
1	9	14								
2	20	17	39							
3	32	19	51	82						
4	34	16	45	80	84					
5	3	6	5	5	4	4				
6	7	7	11	13	12	3	5			
7	3	6	5	5	4	4	3	4		
8	13	12	24	30	27	4	8	4	16	
9	21	13	32	50	49	4	9	4	19	34

Using canonical headings, now we can define *canonical codes* C.

Definition 8 (*Canonical code*). *Let C be any code from $\mathcal{C}$. We say that C is a canonical (i,j,k) code, denoted by $C_{i,j}^{(k)}$ if C has a canonical heading*

$$F(C_{i,j}^{(k)}) = F_{i,j}^{(k)}.$$

Now the important question is *does any code C from $\mathcal{C}$ equivalent to the canonical code $C_{i,j}^{(k)}$* ? The next statement is very important from this point of view.

Proposition 5 (*Computational result*) *Let L_i be the canonical partition and $\mathrm{Stab}_{G_8}(L_i)$ be its stabilizer group in $G_8 = S_8 \rtimes H_{ev}^8$. Let*

$$G_8 \;=\; \bigcup_{j=0}^{n_i} \boldsymbol{g}_j \mathrm{Stab}_{G_8}(L_i), \quad \textit{where} \;\; n_i = [G_8 : \mathrm{Stab}_{G_8}(L_i)]$$

be the left coset decomposition of the group G_8. Then any such coset has the non-trivial intersection with the stabilizer group $\mathrm{Stab}_{G_8}(W_0)$ of the sphere W_0, i.e.

$$|\mathrm{Stab}_{G_8}(W_0) \cap \boldsymbol{g}_j \mathrm{Stab}_{G_8}(L_i)| \;>\; 0.$$

Using this computational result, now we can guarantee that any code from $\mathcal{C}$ is equivalent to some canonical code.

Proposition 6 *Let $C \in \mathcal{C}$ and let $F = F(C)$ be a heading of C. Then C is G-equivalent to the canonical code $C_{i,j}^{(k)} \in \mathcal{C}$ with heading $F_{i,j}^{(k)}$, where $0 \leq i \leq j \leq 9$ and where the permutation π_k is defined by (**??**).*

Proof. Let C be any code from $\mathcal{C}$. Define the following subset of C

$$Y_2 = \bigcup_{s=0}^{7} \boldsymbol{e}_s \times A_r(\boldsymbol{e}_s) \;=\; \{(\boldsymbol{e}_s \,|\, \boldsymbol{y}) :\; \boldsymbol{e}_s \in W_0,\; \boldsymbol{y} \in A_r(\boldsymbol{e}_s)\}, \tag{7}$$

where $A_r(\boldsymbol{e}_s)$, $s = 0, \ldots, 7$ is a partition A_r of E_{ev}^8. Assume that A_r is equivalent to L_j for some j. Thus there exists an element $\boldsymbol{g}_2 \in G_8$ such that $\boldsymbol{g}_2 A_r = L_j$ and in particular

$$\boldsymbol{g}_2 A_r(\boldsymbol{e}_s) \;=\; L_{j,\tau_2^{-1}(s)}. \tag{8}$$

Let 1_8 be the identity element of the group G_8. Applying the element $1_8 \times \boldsymbol{g}_2$ to C, its subset defined (7), and taking into account (8), we have

$$\begin{aligned}(1_8 \times \boldsymbol{g}_2) Y_2 &= (1_8 \times \boldsymbol{g}_2)\Big\{\bigcup_{s=0}^{7} \boldsymbol{e}_s \times A_r(\boldsymbol{e}_s)\Big\} \\ &= \bigcup_{s=0}^{7} \boldsymbol{e}_s \times (\boldsymbol{g}_2 A_r(\boldsymbol{e}_s)) \\ &= \bigcup_{s=0}^{7} \boldsymbol{e}_s \times L_{j,\tau_2^{-1}(s)} \\ &= \bigcup_{s=0}^{7} \boldsymbol{e}_{\tau_2(s)} \times L_{j,s}.\end{aligned}$$

Set $C' = (1 \times \boldsymbol{g}_2)C$, and define the following subset of C'

$$Y_1 = \bigcup_{s=0}^{7} A_l(\boldsymbol{e}_s) \times \boldsymbol{e}_s \;=\; \{(\boldsymbol{y} \,|\, \boldsymbol{e}_s) :\; \boldsymbol{e}_s \in W_0,\; \boldsymbol{y} \in A_l(\boldsymbol{e}_s)\},$$

where $A_l(\boldsymbol{e}_s)$, $s = 0, \ldots, 7$ is a partition A_l of E_{ev}^8. Assume that A_l is equivalent to L_i for some i. By Proposition 5 there exists an element $\boldsymbol{g}_1 \in G_8$ such that $\boldsymbol{g}_1 A_l = L_i$ and $\boldsymbol{g}_1 W_0 = W_0$. In particular

$$\boldsymbol{g}_1 A_l(\boldsymbol{e}_s) \;=\; L_{i,\tau_1^{-1}(s)}. \tag{9}$$

Applying the element $\boldsymbol{g}_1 \times 1_8$ to C', its subset Y_1, and taking into account (9), we have

$$\begin{aligned}(\boldsymbol{g}_1 \times 1_8)Y_1 &= (\boldsymbol{g}_1 \times 1_8)\Big\{\bigcup_{s=0}^{7} A_l(\boldsymbol{e}_s) \times \boldsymbol{e}_s\Big\} \\ &= \bigcup_{s=0}^{7}(\boldsymbol{g}_1 A_l(\boldsymbol{e}_s)) \times \boldsymbol{e}_s \\ &= \bigcup_{s=0}^{7} L_{i,\tau_1^{-1}(s)} \times \boldsymbol{e}_s.\end{aligned}$$

Moreover, we have

$$\begin{aligned}(\boldsymbol{g}_1 \times 1_8)\Big\{\bigcup_{s=0}^{7} \boldsymbol{e}_{\tau_2(s)} \times L_{j,s}\Big\} &= \bigcup_{s=0}^{7} \boldsymbol{g}_1(\boldsymbol{e}_{\tau_2(s)}) \times L_{j,s} \\ &= \bigcup_{s=0}^{7} \boldsymbol{e}_{\tau_3(s)} \times L_{j,s},\end{aligned}$$

for some permutation $\tau_3 \in S_8$. Since $\boldsymbol{e}_{\tau_3(s)} \in L_{i,\tau_1^{-1}(s)}$ we conclude that $\tau_3 = \tau_1^{-1}$. Set $C'' = (\boldsymbol{g}_1 \times 1_8)C'$. Then C'' is equivalent to C and its heading by definition is equal to

$$\bigcup_{s=0}^{7} \boldsymbol{e}_{\tau_1^{-1}(s)} \times L_{j,s} \ \bigcup\ \bigcup_{s=0}^{7} L_{i,\tau_1^{-1}(s)} \times \boldsymbol{e}_s.$$

Without loss of generality we can assume that $i \leq j$ (if not apply the permutation of S_2 from the definition of G, i.e. switch the blocks of coordinates). $\triangle$

It is clear that a code C can have several different headings. Now by lemma 4 we know that the complementary vector $\bar{\boldsymbol{e}}_s$ belongs to the code $A(\boldsymbol{e}_s)$ for all $s = 0, ..., 7$. This gives us 248 vectors of $F_{i,j}^{(k)}$. Now consider how to describe the remaining part of code C. Denote by $\boldsymbol{b}_0, \boldsymbol{b}_1, ..., \boldsymbol{b}_7$ the following vectors, which form, with their complementary vectors $\bar{\boldsymbol{b}}_0, \bar{\boldsymbol{b}}_1, \ldots, \bar{\boldsymbol{b}}_7$, the unique Hamming (linear) $(8, 4, 16)$-code, which we denote V:

$$\begin{aligned}\boldsymbol{b}_0 &= (00000000), & \boldsymbol{b}_1 &= (00001111),\\ \boldsymbol{b}_2 &= (00110011), & \boldsymbol{b}_3 &= (11000011),\\ \boldsymbol{b}_4 &= (01010101), & \boldsymbol{b}_5 &= (10100101),\\ \boldsymbol{b}_6 &= (10011001), & \boldsymbol{b}_7 &= (01101001),\end{aligned}$$

here $\boldsymbol{b}_0 = \boldsymbol{e}_0$. For any vector $\boldsymbol{b}_s$, where $s = 1, ..., 7$ the corresponding ball $W(\boldsymbol{b}_s)$ is mapped to an ordered partition, say $P_s = L(\boldsymbol{b}_s)$ where $P_0 = L_j$ and $P_s = (P_{s,0}, P_{s,1}, \ldots, P_{s,7})$ so that $\boldsymbol{e}_k \in P_{s,k}$, for $k = 0, 1, \ldots, 7$. Observe that for any i,k and s

$$|L_{i,k} \cap W(\boldsymbol{b}_s)| = 1.$$

Thus we have the following general form of any code C from the set $\mathcal{C}$.

Theorem 3 *Let C be any $(16, 4, 2^{11})$ code with rank 14 over $\mathbb{F}_2$. Then*

$$C = \bigcup_{s=0}^{7} \bigcup_{k=0}^{7} \{(\boldsymbol{a} \,|\, \boldsymbol{b}),\ (\bar{\boldsymbol{a}} \,|\, \boldsymbol{b}):\ \boldsymbol{a} \in L_{i,\pi^{-1}(s)} \cap W(\boldsymbol{b}_k),\quad \boldsymbol{b} \in P_{s,k}\}$$

§ 6. Canonical $(16, 4, 2^{11})$-codes of rank 14

Theorem 3 gives the general form of an arbitrary code from $\mathcal{C}$. As we know, any such code by Proposition 6 is equivalent to a canonical one. Our next purpose is to find all different canonical codes. We do it by using special functions where every function represents a canonical code.

Fix a canonical partition L_i $(0 \leq i \leq 9)$ and a permutation $\pi \in S_8$. Define the function $\Lambda_{i,\pi}$: $\{W(\boldsymbol{b})\} \longrightarrow \Omega$, which for each arbitrary vector $\boldsymbol{b}$ gives a partition P from Ω as:

$$\Lambda_{i,\pi}(W(\boldsymbol{b}_s)) \ = \ P \in \Omega, \tag{10}$$

where $P = (P_0, P_1, \ldots, P_7)$ is ordered so that $\boldsymbol{e}_k \in P_k$, for $k = 0, \ldots, 7$ and the value of the function $\Lambda_{i,\pi}$ on elements of the ball $W(\boldsymbol{b}_s)$

$$\Lambda_{i,\pi}(\boldsymbol{a}) \ = \ P_k, \text{ where } \boldsymbol{a} = L_{i,\pi^{-1}(k)} \cap W(\boldsymbol{b}_s). \tag{11}$$

Definition 9 *Fix a canonical partition L_i $(0 \leq i \leq 9)$ and a permutation $\pi \in S_8$. We say that the function $\Lambda = \Lambda_{i,\pi}$ is admissible, if it satisfies:*

(1) $\Lambda(\boldsymbol{b}_0) \ = \ L_j$.

(2) $\Lambda(\boldsymbol{b}_s) \ = \ \Lambda(\bar{\boldsymbol{b}}_s)$.

(3) *For any pair $\boldsymbol{x}, \boldsymbol{y}$, $\boldsymbol{x} \in W(\boldsymbol{b}_j)$, $\boldsymbol{y} \in W(\boldsymbol{b}_k)$, the condition $d(\boldsymbol{x}, \boldsymbol{y}) = 2$ implies that*

$$\Lambda(\boldsymbol{x}) \ \cap \ \Lambda(\boldsymbol{y}) \ = \ \emptyset. \tag{12}$$

Thus all possible admissible functions give all different canonical codes. As it turns out it is not difficult to find all of them.

Lemma 6 (*Computational results*). *For any i and j, where $0 \leq i \leq j \leq 9$ the number of admissible functions* (*canonical codes*) *is given by the following table*

$j \setminus i$	0	1	2	3	4	5	6	7	8	9
0	644									
1	160	176								
2	100	93	125							
3	434	204	215	575						
4	372	120	161	536	500					
5	328	300	48	168	80	2205				
6	64	43	36	61	62	48	25			
7	3	6	5	5	4	4	3	4		
8	216	128	101	319	180	184	62	4	123	
9	180	64	81	204	220	64	33	4	115	113

The total number of distinct canonical $(16, 4, 2^{11})$ *codes in* $\mathcal{C}$ *is equal to* 10312.

Now we formulate several lemmas obtained by the direct computations, which give the distribution of the canonical codes with given size of the kernel. Almost all codes from $\mathcal{C}$ have ranks in the interval $4 \leq \ker(C) \leq 64$. We give the same tables on i and j as above for all values of kernels. Agree that in the all tables which we give below, we omit the zero j-th rows.

Lemma 7 (*Codes with kernel* 4). *The number of admissible functions whose corresponding* *codes have kernel of size* 4 *is given by the following table:*

$j \setminus i$	0	1	2	3	4	5	6	7	8	9
0	130									
3	60	0	0	103						
5	80	0	0	25	0	721				
6	9	0	0	16	0	5	13			
7	0	0	0	0	0	0	3	4		
9	26	0	0	52	0	10	24	4	0	66

The total number of distinct $(16, 4, 2^{11})$ *codes in* $\mathcal{C}$ *with kernel* 4 *is equal to* 1351*.*

Lemma 8 (*Codes with kernel* 8*). The number of admissible functions whose corresponding codes have kernel of size* 8 *is given by the following table:*

$j \setminus i$	0	1	2	3	4	5	6	7	8	9
0	250									
1	58	0								
2	14	0	0							
3	159	100	63	209						
4	99	0	0	219	0					
5	132	90	5	65	20	690				
6	25	35	34	43	56	15	7			
7	0	6	5	5	4	0	0	0		
8	77	0	0	133	0	55	29	0	0	
9	72	59	78	122	194	21	7	0	58	32

The total number of distinct $(16, 4, 2^{11})$ *codes in* $\mathcal{C}$ *with kernel* 8 *is equal to* 3345*.*

Lemma 9 (*Codes with kernel* 16)*. The number of admissible functions whose corresponding codes have kernel of size* 16 *is given by the following table:*

$j \setminus i$	0	1	2	3	4	5	6	7	8	9
0	191									
1	83	138								
2	54	59	60							
3	156	102	150	235						
4	202	103	127	301	426					
5	79	153	24	59	38	617				
6	29	6	1	2	6	25	4			
7	3	0	0	0	0	0	0	0		
8	102	78	39	139	116	92	30	4	65	
9	76	5	3	27	25	29	2	0	55	11

The total number of distinct $(16, 4, 2^{11})$ *codes in* $\mathcal{C}$ *with kernel* 16 *is equal to* 4331*.*

Lemma 10 (*Codes with kernel* 32)*. The number of admissible functions whose corresponding codes have kernel of size* 32 *is given by the following table:*

$j \setminus i$	0	1	2	3	4	5	6	7	8	9
0	61									
1	19	38								
2	32	32	64							
3	56	2	2	26						
4	69	17	34	16	73					
5	32	54	16	16	20	132				
6	1	2	1	0	0	2	1			
7	0	0	0	0	0	4	0	0		
8	28	46	59	46	63	32	2	0	50	
9	6	0	0	3	1	4	0	0	2	4

The total number of distinct $(16,4,2^{11})$ *codes in* $\mathcal{C}$ *with kernel* 32 *is equal to* 1168.

Lemma 11 (*Codes with kernel* 64). *The number of admissible functions whose corresponding codes have kernel of size* 64 *is given by the following table:*

$j \setminus i$	0	1	2	3	4	5	6	7	8	9
0	11									
2	0	2	1							
3	3	0	0	2						
4	2	0	0	0	1					
5	4	3	3	3	2	44				
6	0	0	0	0	0	1	0			
8	9	4	3	1	1	4	1	0	7	

The total number of distinct $(16,4,2^{11})$ *codes in* $\mathcal{C}$ *with kernel* 64 *is equal to* 112.

Lemma 12 (*Codes with kernels* 128 *and* 256). *The number of admissible functions whose corresponding codes have kernel of size* 128 *and* 256 *are given by the values:*

$$\mu(0,0) \;=\; \mu(5,0) \;=\; \mu(8,5) \;=\; \mu(8,8) \;=\; 1.$$

and for the kernel 256

$$\mu(5,5) \;=\; 1.$$

§ 7. Non-equivalent $(16,4,2^{11})$-codes of rank 14

Our next goal is to identify the G-equivalent canonical codes (i.e. equivalence under the action of the group G). It is obvious that codes whose kernels have different size are non-equivalent.

Lemma 13 *Let $C \in \mathcal{C}$ be a code and $F(C)$ its heading. Then for any*

$$\boldsymbol{g} \in \mathrm{Stab}_{S_8 \times S_8}(W(\mathbf{0}) \times W(\mathbf{0})) \;\simeq\; S_7 \times S_7,$$

we have $F(\boldsymbol{g}C) \;=\; \boldsymbol{g}F(C)$.

Proof. Indeed since $\boldsymbol{g}$ stabilizes the ball $W(\mathbf{0})$. $\triangle$

Now we can formulate the criteria of equivalence of two canonical codes. As it turns out, the verification of this criteria consumes the major part of the computational efforts.

Theorem 4 *Two canonical codes $C = C_{i,j}^{(k)}$ and $C' = C_{i',j'}^{(k')}$ are equivalent if and only if*

$$\tau(C + \boldsymbol{h}) \;=\; \boldsymbol{g}C', \tag{13}$$

where $\boldsymbol{h} \in E_{ev}^8 \times E_{ev}^8$, $\boldsymbol{g} \in S_7 \times S_7$ and $\tau \in S_7 \times S_7 \backslash G_8$.

Proof. For given $C = C_{i,j}^{(k)}$ and $C' = C_{i',j'}^{(k')}$, suppose there exists $\pi \in S_{16}$ and $\boldsymbol{h} \in H_{16}$ so that $\pi(C + \boldsymbol{h}) = C'$. Let $\boldsymbol{h}' \in C'$, then adding it to both sides gives

$$\pi(C + \boldsymbol{h}_1) = C'', \quad \text{where} \;\; C'' = C' + \boldsymbol{h}', \quad \text{and} \;\; \boldsymbol{h}_1 = \boldsymbol{h} + \pi^{-1}(\boldsymbol{h}').$$

Note that C'' has zero word and satisfies the parity rule. Consequently $C_1 = C + \boldsymbol{h}_1$ has zero word and satisfies parity rule. Applying lemma 1 it follows that $\pi \in S_2 \rtimes (S_8 \times S_8)$. Since $\boldsymbol{h}' \in C'$ satisfies parity rule, so does $\pi^{-1}(\boldsymbol{h}')$. Thus $\boldsymbol{h}$ satisfies the parity rule, i.e. $\boldsymbol{h} \in E_{\mathrm{ev}}^8 \times E_{\mathrm{ev}}^8$. The permutation π can be written as $\pi = \boldsymbol{g}^{-1}\tau$, where $\boldsymbol{g} \in S_7 \times S_7$. $\triangle$

Thus, in order to find all non-equivalent codes with given kernel, we have to check all pairs of codes C and C' for this possible equality (13). Now we can formulate the main result of the paper which has been obtained by overall checking of all possible pairs of codes with given kernel.

Theorem 5 (*Computational results*) *Among non-equivalent binary extended perfect codes of length $n = 16$ with rank 14 over $\mathbb{F}_2$ there are exactly 1708 codes of rank 14, in particular, 844 Solovieva-Phelps codes and 864 codes, obtained by Etzion-Vardi construction and its generalization. These codes are distributed over the kernel as follows. There are exactly*

– 101 *codes with kernel* 4*;*

– 448 *codes with kernel* 8*;*

– 780 *codes with kernel* 16*;*

– 321 *codes with kernel* 32*;*

– 53 *codes with kernel* 64*;*

– 4 *codes with kernel* 128*;*

– 1 *code with kernel* 256*.*

For comparison we give here also the distribution of Solovieva-Phelps codes of rank 14 over the size of kernel, obtained by Phelps in [6] (see Theorem 2).

Theorem 6 (*Phelps* [6]). *Among non-equivalent Solovieva-Phelps codes of length* $n = 16$ *with rank* 14 *over* $\mathbb{F}_2$ *there are exactly* 844 *codes. These codes are distributed over the kernel as follows:*

– 35 *codes with kernel* 4*;*

– 172 *codes with kernel* 8*;*

– 374 *codes with kernel* 16*;*

– 210 *codes with kernel* 32*;*

– 48 *codes with kernel* 64*;*

– 4 *codes with kernel* 128*;*

– 1 *code with kernel* 256*.*

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