# TWISTED CHARACTER OF A SMALL REPRESENTATION OF GL(4)

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Abstract. We compute by a purely local method the (elliptic) $\theta$-twisted character $\chi_{\pi_Y}$ of the representation $\pi_Y = I_{(3,1)}(1_3 \times \chi_Y)$ of $G = \mathrm{GL}(4, F)$, where $F$ is a $p$-adic field, $p \neq 2$, and $Y$ is an unramified quadratic extension of $F$; $\chi_Y$ is the nontrivial character of $F^\times / N_{Y/F} Y^\times$.

The representation $\pi_Y$ is normalizedly induced from $\begin{pmatrix} m_3 & * \\ 0 & m_1 \end{pmatrix} \mapsto \chi_Y(m_1)$, $m_i \in \mathrm{GL}(i, F)$, on the maximal parabolic subgroup of type $(3,1)$; $\theta$ is the "transpose-inverse" involution of $G$.

We show that the twisted character $\chi_{\pi_Y}$ of $\pi_Y$ is an unstable function: its value at a twisted regular elliptic conjugacy class with norm in $C_Y = \mathbf{C}_Y(F)$="$(\mathrm{GL}(2,Y)/F^\times)_F$" is minus its value at the other class within the twisted stable conjugacy class. It is 0 at the classes without norm in $C_Y$. Moreover $\pi_Y$ is the endoscopic lift of the trivial representation of $C_Y$.

We deal only with unramified $Y/F$, as globally this case occurs almost everywhere. The case of ramified $Y/F$ would require another paper.

Our $\mathbf{C}_Y$ ="$(\mathrm{R}_{Y/F}\,\mathrm{GL}(2)/\,\mathrm{GL}(1))_F$" has $Y$-points $\mathbf{C}_Y(Y) = \{(g, g') \in \mathrm{GL}(2,Y) \times \mathrm{GL}(2,Y);\ \det(g) = \det(g')\}/Y^\times$ ($Y^\times$ embeds diagonally); $\sigma(\neq 1)$ in $\mathrm{Gal}(Y/F)$ acts by $\sigma(g, g') = (\sigma g', \sigma g)$. It is a $\theta$-twisted elliptic endoscopic group of GL(4).

Naturally this computation plays a role in the theory of lifting of $C_Y$ and GSp(2) to GL(4) using the trace formula, to be discussed elsewhere.

Our work extends – to the context of nontrivial central characters – the work of [FZ4], where representations of $\mathrm{PGL}(4, F)$ are studied. In [FZ4] a 4-dimensional analogue of the model of the small representation of $\mathrm{PGL}(3, F)$ introduced with Kazhdan in [FK] in a 3-dimensional case is developed, and the local method of computation introduced in [FZ3] is extended. As in [FZ4] we use here the classification of twisted (stable) regular conjugacy classes in $\mathrm{GL}(4, F)$ of [F], motivated by Weissauer [W].

## Introduction

Let $\pi$ be an admissible representation (see Bernstein-Zelevinsky [BZ], 2.1) of a $p$-adic reductive group $G$. Its character $\chi_\pi$ is a complex valued function defined by $\mathrm{tr}\,\pi(fdg) = \int_G \chi_\pi(g) f(g) dg$ for all complex valued smooth compactly supported measures $fdg$ ([BZ], 2.17). It is smooth on the regular set of the group $G$. The character is important since it characterizes the representation up to equivalence. A fundamental result of Harish-Chandra [H] establishes that the character is a locally integrable function in characteristic zero.

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Let $\theta$ be an automorphism of finite order of the group $G$. Define ${}^\theta\pi$ by ${}^\theta\pi(g) = \pi(\theta(g))$. When $\pi$ is invariant under the action of $\theta$ (thus ${}^\theta\pi$ is equivalent to $\pi$), Shintani and others introduced an extension of $\pi$ to the semidirect product $G \rtimes \langle\theta\rangle$. The twisted character $\chi_\pi(g \times \theta)$ is defined by $\operatorname{tr}\pi(fdg \times \theta) = \int_G \chi_\pi(g \times \theta) f(g)dg$ for all $fdg$. It depends only on the $\theta$-conjugacy class $\{hg\theta(h)^{-1}; h \in G\}$ of $g$. It is again smooth on the $\theta$-regular set, and characterizes the $\theta$-invariant irreducible $\pi$ up to isomorphism. Moreover, it is locally integrable (see Clozel [C]) in characteristic zero.

Characters provide a very precise tool to express a relation of representations of different groups, called lifting, initiated by Shintani and studied extensively in the case of base change, and also in non base change situations such as twisting by characters (Kazhdan [K], Waldspurger [Wa]), and the symmetric square lifting from SL(2) to PGL(3) ([Fsym], [FK]). In this last case twisted characters of $\theta$-invariant representations of PGL(3) are related to packets of representations of SL(2), and $\theta$ is the involution sending $g$ to its transpose-inverse.

*The aim of the present work is to compute the twisted (by $\theta$) character of a specific representation $\pi = \pi_Y = I_{(3,1)}(1_3 \times \chi_Y)$, of the group $G = \mathrm{GL}(4, F)$, $F$ a p-adic field, p odd.* Here $Y/F$ is an unramified quadratic extension and $\chi_Y$ is the quadratic character of $F^\times$ which is trivial on the group $N_{Y/F}Y^\times$, where $N_{Y/F}$ is the norm map from $Y$ to $F$. This $\pi$ is normalizedly induced from the representation $\left(\begin{smallmatrix} m_3 & * \\ 0 & m_1 \end{smallmatrix}\right) \mapsto \chi_Y(m_1)$, $m_i \in \mathrm{GL}(i, F)$, of the standard (upper triangular) maximal parabolic subgroup $P$ of type $(3, 1)$. It is invariant under the involution $\theta(g) = J^{-1}{}^t g^{-1} J$, where $J = (a_i\delta_{i,5-j})$, $a_1 = a_2 = 1$, $a_3 = a_4 = -1$.

A natural setting for the statement of our result is the theory of liftings to the group $\mathbf{G} = \mathrm{GL}(4)$ from its $\theta$-twisted endoscopic (see Kottwitz-Shelstad [KS]) $F$-group

$$\mathbf{C}_Y = \{(g, g') \in \mathrm{GL}(2) \times \mathrm{GL}(2); \det g = \det g'\}/\mathbb{G}_m,$$

where the multiplicative group $\mathbb{G}_m = \mathrm{GL}(1)$ embeds as $z \mapsto (zI_2, zI_2)$, $I_2$ is the identity $2 \times 2$ matrix, with $\mathrm{Gal}(\overline{F}/F)$-action which is a composition of the usual Galois action on each of the two factors GL(2) with the transposition $(g, g') \mapsto (g', g)$ if $\sigma \in \mathrm{Gal}(\overline{F}/F)$ has nontrivial restriction to $Y$. Here $\overline{F}$ is a separable algebraic closure of $F$ containing $Y$.

The corresponding map $\lambda_Y$ of dual groups is simply the natural embedding in $\hat{G} = \mathrm{GL}(4, \mathbb{C})$ of the non connected $\hat{C}_Y = Z_{\hat{G}}(\hat{s}\hat{\theta})$

$$= \left\{ g \in \hat{G} = \mathrm{GL}(4, \mathbb{C});\ g\hat{s}J{}^t g = \hat{s}J = \begin{pmatrix} 0 & \omega \\ \omega^{-1} & 0 \end{pmatrix} \right\} = \mathrm{O}\left( \begin{pmatrix} 0 & \omega \\ \omega^{-1} & 0 \end{pmatrix}, \mathbb{C} \right)$$

$$= \left\langle \begin{pmatrix} aB & bB \\ cB & dB \end{pmatrix}, \iota = \begin{pmatrix} 1 & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ 0 & & & 1 \end{pmatrix};\ \left(A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, B\right) \in \left\{ \mathrm{GL}(2, \mathbb{C})^2; \det A \cdot \det B = 1 \right\} / \mathbb{C}^\times \right\rangle,$$

where $z \in \mathbb{C}^\times$ embeds as the central element $(z, z^{-1})$, and where $\hat{s} = \mathrm{diag}(-1, 1, -1, 1)$ and $\omega = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Thus $\hat{C}_Y$ is the $\hat{\theta}$-centralizer in $\hat{G}$ of the semisimple element $\hat{s}$ (and $\hat{\theta}$ is defined on $\hat{G}$ by the same formula that defines $\theta$ on $G$), and $\mathrm{Gal}(Y/F)$ acts via conjugation by $\iota$.

Indeed, our result can be viewed as asserting that the $\theta$-invariant representation $\pi$ of $G = \mathbf{G}(F)$, whose central character is $\chi_Y \neq 1$ of order two, is the endoscopic lift of the trivial representation of $C_Y = \mathbf{C}_Y(F) = (\mathrm{GL}(2,Y)/F^\times)_F$. The subscript $F$ here indicates that $(\mathrm{GL}(2,Y)/F^\times)_F$ consists of $g$ in $\mathrm{GL}(2,Y)/F^\times$ with $\det(g)$ in $F^\times/F^{\times 2}$.

To state this we note that the embedding $\lambda_Y : \hat{C}_Y \to \hat{G}$ defines a norm map. This norm map relates the stable $\theta$-conjugacy classes in $G$ with stable conjugacy classes in $C_Y$, where "stable" means the elements in $G$ of an orbit in the points of $\mathbf{G}$ in a separable algebraic closure $\overline{F}$ of $F$. The crucial case is that of $\theta$-elliptic elements. A stable $\theta$-conjugacy class consists of several $\theta$-conjugacy classes. The stable $\theta$-conjugacy classes of elements in $G$, and the $\theta$-conjugacy classes within the stable $\theta$-classes, have been described recently in [F], in analogy with the description of the (stable) conjugacy classes in the group of symplectic similitudes $\mathrm{GSp}(2,F)$ of Weissauer [W]. In fact in [FZ4] the simpler case of $\mathrm{PGL}(4,F)$ is used, but here, as in [F], we deal with $\theta$-classes in $\mathrm{GL}(4,F)$. We give here full details of the description in our case.

There are four types of $\theta$-elliptic elements of $G$, named in [F] and here I, II, III, IV, depending on their splitting behaviour. As in [F], our work relies on an explicit presentation of representatives of the $\theta$-conjugacy classes within the stable such classes in $G$. We present here the same set of representatives as in [FZ4].

The norm map, which we describe explicitly here, relates $\theta$-conjugacy classes of types II and IV to conjugacy classes in $C_Y$. It does not relate classes of types I, III to classes in $C_Y$. Our "quadratic" case behaves then in a complementary fashion to that of [FZ4], where $\theta$-conjugacy classes of types I, III are related to conjugacy classes in the group $C = \mathrm{SO}(4)$ of [FZ4], but $\theta$-conjugacy classes of types II, IV are not related to conjugacy classes in $C$.

*We prove that the $\theta$-character of $\pi = \pi_Y$, $\chi_\pi(g \times \theta)$, vanishes on $\theta$-regular elements $g$ of type I and III.* The stable $\theta$-conjugacy classes of types II and IV come associated with a quadratic extension $E/E_3$, where $Y = E_3$ is a quadratic extension of $F$. The two $\theta$-conjugacy classes $g_r$ within the stable $\theta$-classes are parametrized by $r$ in $E_3^\times/N_{E/E_3}E^\times$. *We show that the value of $\chi_\pi(g_r \times \theta)$, multiplied by a suitable Jacobian $\Delta(g_r\theta)/\Delta_C(Ng)$, is $2\kappa(r)$; here $\kappa$ is the nontrivial character of $E_3^\times/N_{E/E_3}E^\times$.* We deal only with unramified $Y/F$, as globally this case occurs almost everywhere. The case of ramified $Y/F$ would require another paper.

In particular the character $\chi_\pi(g \times \theta)$ is an unstable function, namely its value at one $\theta$-conjugacy class within a stable $\theta$-conjugacy class of type II or IV is negative its value at the other $\theta$-conjugacy class.

Our result is a special case of the lifting with respect to $\lambda_Y$ to the group $G = \mathrm{GL}(4,F)$ of representations of the group $C_Y = (\mathrm{GL}(2,Y)/F^\times)_F$.

Our work develops the method of [FZ4] to the context of representations with nontrivial central characters. We use a model of our representation $\pi_Y = I_{(3,1)}(1_3 \times \chi_Y)$, different from the standard model of a parabolically induced representation. It is a twist of the four dimensional analogue of [FZ4], dealing with $\pi_4 = I_{(3,1)}(1_3 \times 1)$, of a (three dimensional) model introduced and used with Kazhdan in [FK] to compute the twisted (by transpose-inverse) character of the representation $\pi_3 = I_{(2,1)}(1_2)$ of $\mathrm{PGL}(3,F)$ normalizedly induced from the trivial representation of the maximal parabolic subgroup. We do not use our results to prove the fundamental lemma since in our case, as well as that of [FZ4], the fundamental lemma is already established in [F]. In the case of the symmetric square lifting from $\mathrm{SL}(2,F)$ to $\mathrm{PGL}(3,F)$, an analogous purely local and simple proof of the fundamental lemma was given in [Fsym. Unit elements].

The work of [FK] uses local arguments to compute the twisted character of $\pi_3$ on one of the two twisted conjugacy classes within the stable one (where the quadratic form is anisotropic), and global arguments to reduce the computation on the other class (where the quadratic form is isotropic) to that computed by local means. A purely local computation for the second class is given in [FZ3]. In [FZ4] this local computation is developed in a four dimensional projective case. A global type of argument as in [FK] is harder to apply as there are not enough anisotropic quadratic forms in the four dimensional case. Anyway, a simpler, local proof, is better. Here we extend the work of [FZ4] to $\theta$-invariant representations of $\mathrm{GL}(4,F)$ whose central character is nontrivial, necessarily quadratic. Our work is parallel to – but entirely independent of – the work of [FZ4].

## Conjugacy classes

Let $F$ be a local nonarchimedean field, and $R$ its ring of integers. Put $\mathbf{G}=\mathrm{GL}(4)$, $G=\mathbf{G}(F)$ and $K=\mathbf{G}(R)$. Put $\mathbf{C}_Y=\{(g_1,g_2)\in\mathrm{GL}(2)\times\mathrm{GL}(2);\ \det(g_1)=\det(g_2)\}/\mathbb{G}_m$ ($\mathbb{G}_m$ embeds diagonally), viewed as a group over $F$ with Galois action $\tau(g,g')=(\tau g,\tau g')$ unless $\tau\in\mathrm{Gal}(\overline{F}/F)$ has nontrivial restriction to $Y$, in which case $\tau(g,g')=(\tau g',\tau g)$, where $\tau(g_{ij})=(\tau g_{ij})$. It is a form of the group $\mathbf{C}$ of [FZ4], and in particular $\mathbf{C}_Y(Y)=\mathbf{C}(Y)$, but the $\mathrm{Gal}(\overline{F}/F)$-action is different: $\tau\in\mathrm{Gal}(\overline{F}/F)$ takes $(g,g')$ of $\mathbf{C}(\overline{F})$ to $(\tau g,\tau g')$. Then $C_Y=\mathbf{C}_Y(F)=\{g\in GL(2,Y)/F^\times;\ \det(g)\in F^\times\}$ and $K_{C_Y}=\mathbf{C}_Y(R)$. Set $\theta(\delta)=J^{-1}{}^t\delta^{-1}J$ for $\delta$ in $G$. Here $J$ is $\begin{pmatrix}0&w\\-w&0\end{pmatrix}$, where $w=\begin{pmatrix}0&1\\1&0\end{pmatrix}$. Fix a separable algebraic closure $\overline{F}$ of $F$. The elements $\delta$, $\delta'$ of $G$ are called (*stably*) *$\theta$-conjugate* if there is $g$ in $G$ (resp. $\mathrm{GL}(4,\overline{F})$) with $\delta'=g^{-1}\delta\theta(g)$.

We recall some results of [F] concerning (stable) $\theta$-twisted regular conjugacy classes. There are four types of $\theta$-elliptic classes, but the norm map $N$ from $G$ to $C_Y$ relates only the twisted classes in $G$ of type II and IV to conjugacy classes in $C_Y$. We should then expect the twisted character of the representation considered here to vanish on the twisted classes of type I and III.

A set of representatives for the $\theta$-conjugacy classes within a stable semisimple $\theta$-conjugacy class of type I in $\mathrm{GL}(4,F)$ which splits over a quadratic extension $E=F(\sqrt{D})$ of $F$, $D\in F-F^2$, is parametrized by $(\mathbf{r},\mathbf{s})\in F^\times/N_{E/F}E^\times\times F^\times/N_{E/F}E^\times$ ([F], p. 16). Representatives for the $\theta$-regular (thus $t\theta(t)$ is regular) stable $\theta$-conjugacy classes of type I in $\mathrm{GL}(4,F)$ which split over $E$ can be found in a torus $T=\mathbf{T}(F)$, $\mathbf{T}=h^{-1}\mathbf{T}^*h$, $\mathbf{T}^*$ denoting the diagonal subgroup in $\mathbf{G}$, $h=\theta(h)$, and

$$T=\left\{t=\begin{pmatrix}a_1&0&0&a_2D\\0&b_1&b_2D&0\\0&b_2&b_1&0\\a_2&0&0&a_1\end{pmatrix}=h^{-1}t^*h;\quad t^*=\mathrm{diag}(a,b,\sigma b,\sigma a)\in T^*\right\}.$$

Here $a=a_1+a_2\sqrt{D}$, $b=b_1+b_2\sqrt{D}\in E^\times$, and $t$ is regular if $a/\sigma a$ and $b/\sigma b$ are distinct and not equal to $\pm 1$. Note that here $T^* = \mathbf{T}^*(F)$ where the Galois action is that obtained from the Galois action on $T$.

A set of representatives for the $\theta$-conjugacy classes within a stable $\theta$-conjugacy class can be chosen in $T$. Indeed, if $t = h^{-1}t^*h$ and $t_1 = h^{-1}t_1^*h$ in $T$ are stably $\theta$-conjugate, then there is $g = h^{-1}\mu h$ with $t_1 = gt\theta(g)^{-1}$, thus $t_1^* = \mu t^*\theta(\mu)^{-1}$ and $t_1^*\theta(t_1^*) = \mu t^*\theta(t^*)\mu^{-1}$. Since $t$ is $\theta$-regular, $\mu$ lies in the $\theta$-normalizer of $\mathbf{T}^*(\overline{F})$ in $\mathbf{G}(\overline{F})$. Since the group $W^\theta(\mathbf{T}^*, \mathbf{G}) = N^\theta(\mathbf{T}^*, \mathbf{G})/\mathbf{T}^*$, quotient by $\mathbf{T}^*(\overline{F})$ of the $\theta$-normalizer of $\mathbf{T}^*(\overline{F})$ in $\mathbf{G}(\overline{F})$, is represented by the group $W^\theta(T^*, G) = N^\theta(T^*, G)/T^*$, quotient by $T^*$ of the $\theta$-normalizer of $T^*$ in $G$, we may modify $\mu$ by an element of $W^\theta(T^*, G)$, that is replace $t_1$ by a $\theta$-conjugate element, and assume that $\mu$ lies in $\mathbf{T}^*(\overline{F})$. In this case $\mu\theta(\mu)^{-1} = \mathrm{diag}(u, u', \sigma u', \sigma u)$ (since $t$, $t_1$ lie in $T^*$), with $u = \sigma u$, $u' = \sigma u'$ in $F^\times$. Such $t$, $t_1$ are $\theta$-conjugate if $g \in G$, thus $g \in T$, so $\mu = \mathrm{diag}(v, v', \sigma v', \sigma v) \in T^*$ and $\mu\theta(\mu)^{-1} = \mathrm{diag}(v\sigma v, v'\sigma v', v'\sigma v', v\sigma v)$. Hence a set of representatives for the $\theta$-conjugacy classes within the stable $\theta$-conjugacy class of the $\theta$-regular $t$ in $T$ is given by $t \cdot \mathrm{diag}(\mathbf{r}, \mathbf{s}, \mathbf{s}, \mathbf{r})$, where $\mathbf{r}$, $\mathbf{s} \in F^\times/N_{E/F}E^\times$.

A set of representatives for the $\theta$-conjugacy classes within a stable semisimple $\theta$-conjugacy class of type II in $\mathrm{GL}(4, F)$ which splits over the biquadratic extension $E = E_1E_2$ of $F$ with Galois group $\langle\sigma, \tau\rangle$, where $E_1 = F(\sqrt{D}) = E^\tau$, $E_2 = F(\sqrt{AD}) = E^{\sigma\tau}$, $E_3 = F(\sqrt{A}) = E^\sigma$ are quadratic extensions of $F$, thus $A, D \in F - F^2$, is parametrized by $\mathbf{r} \in F^\times/N_{E_1/F}E_1^\times$, $\mathbf{s} \in F^\times/N_{E_2/F}E_2^\times$ ([F], p. 16). It is given by

$$\begin{pmatrix} a_1\mathbf{r} & 0 & 0 & a_2D\mathbf{r} \\ 0 & b_1\mathbf{s} & b_2AD\mathbf{s} & 0 \\ 0 & b_2\mathbf{s} & b_1\mathbf{s} & 0 \\ a_2\mathbf{r} & 0 & 0 & a_1\mathbf{r} \end{pmatrix} = h^{-1}t^*h \cdot \mathrm{diag}(\mathbf{r}, \mathbf{s}, \mathbf{s}, \mathbf{r}), \qquad t^* = \mathrm{diag}(a, b, \tau b, \sigma a).$$

Here $a = a_1 + a_2\sqrt{D} \in E_1^\times$, $b = b_1 + b_2\sqrt{AD} \in E_2^\times$, $\theta(h) = h$.

A set of representatives for the $\theta$-conjugacy classes within a stable semisimple $\theta$-conjugacy class of type III in $\mathrm{GL}(4, F)$ which splits over the biquadratic extension $E = E_1E_2$ of $F$ with Galois group $\langle\sigma, \tau\rangle$, where $E_1 = F(\sqrt{D}) = E^\tau$, $E_2 = F(\sqrt{AD}) = E^{\sigma\tau}$, $E_3 = F(\sqrt{A}) = E^\sigma$ are quadratic extensions of $F$, thus $A, D \in F - F^2$, is parametrized by $r (= r_1 + r_2\sqrt{A}) \in E_3^\times/N_{E/E_3}E^\times$ ([F], p. 16). Representatives for the stable regular $\theta$-conjugacy classes can be taken in the torus $T = h^{-1}T^*h$, consisting of

$$t = \begin{pmatrix} \boldsymbol{a} & \boldsymbol{b}D \\ \boldsymbol{b} & \boldsymbol{a} \end{pmatrix} = h^{-1}t^*h, \qquad t^* = \mathrm{diag}(\alpha, \tau\alpha, \sigma\tau\alpha, \sigma\alpha),$$

where $h = \theta(h)$ is described in [F], p. 16. This $t$ is $\theta$-regular when $\alpha/\sigma\alpha$, $\tau(\alpha/\sigma\alpha)$ are distinct and $\neq \pm 1$. Here

$$\boldsymbol{a} = \begin{pmatrix} a_1 & a_2A \\ a_2 & a_1 \end{pmatrix}, \qquad \boldsymbol{b} = \begin{pmatrix} b_1 & b_2A \\ b_2 & b_1 \end{pmatrix}; \qquad \text{put also} \qquad \boldsymbol{r} = \begin{pmatrix} r_1 & r_2A \\ r_2 & r_1 \end{pmatrix}.$$

Further $\alpha = a + b\sqrt{D} \in E^\times$, $a = a_1 + a_2\sqrt{A} \in E_3^\times$, $b = b_1 + b_2\sqrt{A} \in E_3^\times$, $\sigma\alpha = a - b\sqrt{D}$, $\tau\alpha = \tau a + \tau b\sqrt{D}$. Representatives for all $\theta$-conjugacy classes within the stable $\theta$-conjugacy class of $t$ can be taken in $T$. In fact if $t' = gt\theta(g)^{-1}$ lies in $T$ and $g = h^{-1}\mu h$, $\mu \in \mathbf{T}^*(\overline{F})$, then $\mu\theta(\mu)^{-1} = \operatorname{diag}(u, \tau u, \sigma\tau u, \sigma u)$ has $u = \sigma u$, thus $u \in E_3^\times$. If $g \in T$, thus $\mu \in T^*$, then $\mu = \operatorname{diag}(v, \tau v, \sigma\tau v, \sigma v)$ and $\mu\theta(\mu)^{-1} = \operatorname{diag}(v\sigma v, \tau v\sigma\tau v, \tau v\sigma\tau v, v\sigma v)$, with $v\sigma v \in N_{E/E_3}E^\times$. We conclude that a set of representatives for the $\theta$-conjugacy classes within the stable $\theta$-conjugacy class of $t$ is given by $t \cdot \operatorname{diag}(\boldsymbol{r}, \boldsymbol{r})$, $r \in E_3^\times / N_{E/E_3}E^\times$.

Representatives for the stable regular $\theta$-conjugacy classes of type IV can be taken in the torus $T = h^{-1}T^*h$, consisting of

$$t = \begin{pmatrix} \boldsymbol{a} & \boldsymbol{bD} \\ \boldsymbol{b} & \boldsymbol{a} \end{pmatrix} = h^{-1}t^*h, \qquad t^* = \operatorname{diag}(\alpha, \sigma\alpha, \sigma^3\alpha, \sigma^2\alpha),$$

where $h = \theta(h)$ is described in [F], p. 18. Here $\alpha$ ranges over a quadratic extension $E = F(\sqrt{D}) = E_3(\sqrt{D})$ of a quadratic extension $E_3 = F(\sqrt{A})$ of $F$. Thus $A \in F - F^2$, $D = d_1 + d_2\sqrt{A}$ lies in $E_3 - E_3^2$ where $d_i \in F$. The normal closure $E'$ of $E$ over $F$ is $E$ if $E/F$ is cyclic with Galois group $\mathbb{Z}/4$, or a quadratic extension of $E$, generated by a fourth root of unity $\zeta$, in which case the Galois group is the dihedral group $D_4$. In both cases the Galois group contains an element $\sigma$ with $\sigma\sqrt{A} = -\sqrt{A}$, $\sigma\sqrt{D} = \sqrt{\sigma D}$, $\sigma^2\sqrt{D} = -\sqrt{D}$. In the $D_4$ case $\operatorname{Gal}(E'/F)$ contains also $\tau$ with $\tau\zeta = -\zeta$, we may choose $D = \sqrt{A}$, $\tau D = D$ and $\sigma\sqrt{D} = \zeta\sqrt{D}$.

In any case, $t$ is $\theta$-regular when $\alpha \neq \sigma^2\alpha$. We write $\alpha = a + b\sqrt{D} \in E^\times$, $a = a_1 + a_2\sqrt{A} \in E_3^\times$, $b = b_1 + b_2\sqrt{A} \in E_3^\times$, $\sigma\alpha = \sigma a + \sigma b\sqrt{\sigma D}$, $\sigma^2\alpha = a - b\sqrt{D}$. Also

$$\boldsymbol{a} = \begin{pmatrix} a_1 & a_2A \\ a_2 & a_1 \end{pmatrix}, \qquad \boldsymbol{b} = \begin{pmatrix} b_1 & b_2A \\ b_2 & b_1 \end{pmatrix}, \qquad \boldsymbol{D} = \begin{pmatrix} d_1 & d_2A \\ d_2 & d_1 \end{pmatrix}.$$

Representatives for all $\theta$-conjugacy classes within the stable $\theta$-conjugacy class of $t$ can be taken in $T$. In fact if $t' = gt\theta(g)^{-1}$ lies in $T$ and $g = h^{-1}\mu h$, $\mu \in \mathbf{T}^*(\overline{F})$, then $\mu\theta(\mu)^{-1} = \operatorname{diag}(u, \sigma u, \sigma^3 u, \sigma^2 u)$ has $u = \sigma^2 u$, thus $u \in E_3^\times$. If $g \in T$, thus $\mu \in T^*$, then $\mu = \operatorname{diag}(v, \sigma v, \sigma^3 v, \sigma^2 v)$ and $\mu\theta(\mu)^{-1} = \operatorname{diag}(v\sigma^2 v, \sigma(v\sigma^2 v), \sigma(v\sigma^2 v), v\sigma^2 v)$, with $v\sigma v \in N_{E/E_3}E^\times$. It follows that a set of representatives for the $\theta$-conjugacy classes within the stable $\theta$-conjugacy class of $t = h^{-1}t^*h = \begin{pmatrix} \boldsymbol{a} & \boldsymbol{bD} \\ \boldsymbol{b} & \boldsymbol{a} \end{pmatrix}$, where $t^* = \operatorname{diag}(\alpha, \sigma\alpha, \sigma^3\alpha, \sigma^2\alpha)$, is given by multiplying $\alpha$ by $r$, that is $t^*$ by $t_0^* = \operatorname{diag}(r, \sigma r, \sigma^3 r, \sigma^2 r)$, where $r = \sigma^2 r$ ranges over a set of representatives for $E_3^\times / N_{E/E_3}E^\times$. Now $t_0 = h^{-1}t_0^*h = \begin{pmatrix} \boldsymbol{r} & \\ & \boldsymbol{r} \end{pmatrix}$. Hence a set of representatives is given by $t \cdot \operatorname{diag}(\boldsymbol{r}, \boldsymbol{r})$, $r \in E_3^\times / N_{E/E_3}E^\times$.

## Norm map

The norm map $N : \mathbf{G} \to \mathbf{C}_Y$ is defined on the diagonal torus $\mathbf{T}^*$ of $\mathbf{G}$ by

$$N(\operatorname{diag}(a, b, c, d)) = (\operatorname{diag}(ab, cd), \operatorname{diag}(ac, bd)).$$

Since both components have determinant $abcd$, the image of $N$ is indeed in $\mathbf{C}_Y$.

In type II we have $a \in E_1^\times$, $E_1 = E^\tau = F(\sqrt{D})$, $b \in E_2^\times$, $E_2 = E^{\sigma\tau} = F(\sqrt{AD})$, and the norm map becomes

$$N(\operatorname{diag}(a, b, \tau b, \sigma a)) = (\operatorname{diag}(ab, \tau(b)\sigma(a)), \operatorname{diag}(a\tau(b), b\sigma(a))).$$

The two components on the right are mapped to each other by $\tau$, while the pairs of eigenvalues ($\{ab, \tau(b)\sigma(a)\}$ and $\{a\tau(b), b\sigma(a)\}$) are permuted by $\sigma$. Hence the right side defines a conjugacy class in $\mathrm{GL}(2, E_3)_F$ (the determinant $ab \cdot \tau(b)\sigma(a) = a\tau(b) \cdot b\sigma(a)$ lies in $F^\times$, and $E_3$ is the fixed field of $\sigma$ in $E$). We choose $Y$ to be the quadratic extension $E_3$ of $F$. The image of this torus is the torus (up to conjugacy) in $C_Y(F)$ which splits over the biquadratic extension $E$ of $F$.

In type IV we have $\alpha \in E^\times$, $E = E_3(\sqrt{D})$, $\alpha\sigma^2\alpha \in E_3^\times$, $E_3 = F(\sqrt{A})$, and the norm map becomes

$$N(\mathrm{diag}(\alpha, \sigma\alpha, \sigma^3\alpha, \sigma^2\alpha)) = (\mathrm{diag}(\alpha\sigma\alpha, \sigma^2\alpha\sigma^3\alpha), \mathrm{diag}(\alpha\sigma^3\alpha, \sigma\alpha\sigma^2\alpha)).$$

Here $\sigma^3$ permutes the two diagonal matrices on the right, and $\sigma^2$ permutes each pair of eigenvalues. Since both components of $N(*)$ have equal determinants in $F^\times$, $\mathrm{diag}(*, *)$ defines a conjugacy class in $\mathrm{GL}(2, E_3)_F$. Hence the norm map defines a conjugacy class in $C_Y = \mathbf{C}_Y(F)$ for each $\theta$-stable conjugacy class of type IV in $G = \mathbf{G}(F)$, where we take $Y$ to be $E_3$.

In types I and III the image of the map $N$ does not correspond to any conjugacy class in $C_Y$, for any quadratic extension $Y$ of $F$.

## Jacobians

The character relation that we study relates the product of the value at $t$ of the twisted character of our representation $\pi_Y = I_{(3,1)}(1_3 \times \chi_Y)$ by a factor $\Delta(t \times \theta)$, with the product by a factor $\Delta_C(Nt)$ of the value at $Nt$ of the character of the (trivial) representation $\mathbf{1}_{C_Y}$ of $C_Y$ which lifts to $\pi_Y$.

The factor $\Delta(t \times \theta)$ is defined by

$$\Delta(t \times \theta)^2 = |\det(1 - \mathrm{Ad}(t\theta))|\,\mathrm{Lie}(G/T)|.$$

Here $t$ lies in the $\theta$-invariant torus $T$ which we take to have the form $T = h^{-1}T^*h$, $T^*$ is the diagonal subgroup and $h = \theta(h)$. Thus in the formula for $\Delta(t \times \theta)$ we may replace $t = h^{-1}t^*h$ and $T$ by the diagonal $t^*$ and $T^*$. Note that $\mathrm{Lie}(G/T^*) = \mathrm{Lie}\,U \oplus \mathrm{Lie}\,U^-$, and the upper and lower triangular subgroups $U$, $U^-$ are $\theta$-invariant. We have

$$|\det(1 - \mathrm{Ad}(t\theta))|\,\mathrm{Lie}\,U| = |\prod_{\Theta}(1 - \sum_{\beta\in\Theta}\beta(t))|,$$

where the product ranges over the orbits $\Theta$ of $\theta$ in the set of positive roots $\beta > 0$, and the sum ranges over the roots in the $\theta$-orbit. Thus for $t = \mathrm{diag}(a, b, c, d)$ we obtain

$$\left|\left(1 - \frac{a}{b}\frac{c}{d}\right)\left(1 - \frac{a}{c}\frac{b}{d}\right)\left(1 - \frac{a}{d}\right)\left(1 - \frac{b}{c}\right)\right|.$$

Further,

$$|\det(1 - \mathrm{Ad}(t\theta))|\,\mathrm{Lie}\,U^-| = \delta(a\theta)^{-1}|\det(1 - \mathrm{Ad}(t\theta))|\,\mathrm{Lie}\,U|$$

where

$$\delta(t\theta) = \prod_{\Theta}((\sum_{\beta\in\Theta}\beta)(t)) = (\frac{a}{b}\frac{c}{d})(\frac{a}{c}\frac{b}{d})(\frac{b}{c})(\frac{a}{d}) = \prod_{\beta>0}\beta(t) = \delta(t).$$

Altogether

$$\Delta(t\theta) = \left|\frac{(ac-bd)^2}{abcd}\cdot\frac{(ab-cd)^2}{abcd}\cdot\frac{(a-d)^2}{ad}\frac{(b-c)^2}{bc}\right|^{1/2}.$$

Similarly,

$$\Delta_C(Nt) = \delta_C^{-1/2}(Nt)|\det(1-Nt)|\operatorname{Lie}U_C| = \left|\frac{ab}{cd}\cdot\frac{ac}{bd}\right|^{-1/2}\left|\left(1-\frac{ab}{cd}\right)\left(1-\frac{ac}{bd}\right)\right|,$$

and so

$$\frac{\Delta(t\theta)}{\Delta_C(Nt)} = \left|\frac{(a-d)^2}{ad}\cdot\frac{(b-c)^2}{bc}\right|^{1/2}.$$

Then in case II if $t = \operatorname{diag}(a, b, \tau b, \sigma a)$, $a = a_1 + a_2\sqrt{D} \in E_1^\times$, $b = b_1 + b_2\sqrt{AD} \in E_2^\times$, we get

$$\frac{\Delta(t\theta)}{\Delta_C(Nt)} = \left|\frac{(a-\sigma a)^2}{a\sigma a}\cdot\frac{(b-\sigma b)^2}{b\sigma b}\right|^{1/2} = \left|\frac{(2a_2\sqrt{D})^2}{a_1^2-a_2^2D}\cdot\frac{(2b_2\sqrt{AD})^2}{b_1^2-b_2^2AD}\right|^{1/2}.$$

In case IV, if $t = \operatorname{diag}(\alpha, \sigma\alpha, \sigma^3\alpha, \sigma^2\alpha)$, $\alpha = a + b\sqrt{D}$, $a = a_1 + a_2\sqrt{A}$, $b = b_1 + b_2\sqrt{A}$, $\sigma\alpha = \sigma a + \sigma b\sqrt{\sigma D}$, $\sigma^3\alpha = \sigma a - \sigma b\sqrt{\sigma D}$, $\alpha - \sigma^2\alpha = 2b\sqrt{D}$, $\sigma(\alpha - \sigma^2\alpha) = 2\sigma b\sqrt{\sigma D}$, and

$$\frac{\Delta(t\theta)}{\Delta_C(Nt)} = \left|\frac{(\alpha-\sigma^2\alpha)^2}{\alpha\sigma^2\alpha}\cdot\frac{\sigma(\alpha-\sigma^2\alpha)^2}{\sigma\alpha\sigma^3\alpha}\right|^{1/2} = \left|\frac{(4b\sigma b)^2D\sigma D}{(a^2-b^2D)\sigma(a^2-b^2D)}\right|^{1/2}.$$

## Characters

Denote by $f$ (resp. $f_{C_Y}$) a complex-valued compactly-supported smooth (thus locally-constant since $F$ is nonarchimedean) function on $G$ (resp. $C_Y$). Fix Haar measures on $G$ and on $C_Y$.

By a $G$-module $\pi$ (resp. $C_Y$-module $\pi_{C_Y}$) we mean an admissible representation ([BZ]) of $G$ (resp. $C_Y$) in a complex space. An irreducible $G$-module $\pi$ is called *$\theta$-invariant* if it is equivalent to the $G$-module ${}^\theta\pi$, defined by ${}^\theta\pi(g) = \pi(\theta(g))$. In this case there is an intertwining operator $A$ on the space of $\pi$ with $\pi(g)A = A\pi(\theta(g))$ for all $g$. Since $\theta^2 = 1$ we have $\pi(g)A^2 = A^2\pi(g)$ for all $g$, and since $\pi$ is irreducible $A^2$ is a scalar by Schur's lemma. We choose $A$ with $A^2 = 1$. This determines $A$ up to a sign. When $\pi$ has a Whittaker model, which happens for all components of cuspidal automorphic representations of the adele group $\mathrm{GL}(4, \mathbb{A})$, we specify a normalization of $A$ which is compatible with a global normalization, as follows, and then put $\pi(g \times \theta) = \pi(g) \times A$.

Fix a nontrivial character $\boldsymbol{\psi}$ of $F$ in $\mathbb{C}^\times$, and a character $\psi(u) = \boldsymbol{\psi}(a_{1,2} + a_{2,3} - a_{3,4})$ of $u = (u_{i,j})$ in the upper triangular subgroup $U$ of $G$. Note that $\psi(\theta(u)) = \psi(u)$. Assume that $\pi$ is a *non degenerate* $G$-module, namely it embeds in the space of "Whittaker" functions $W$ on $G$, which satisfy – by definition – $W(ugk) = \psi(u)W(g)$ for all $g \in G$, $u \in U$, $k$ in a compact open subgroup $K_W$ of $K$, as a $G$-module under right shifts: $(\pi(g)W)(h) = W(hg)$. Then ${}^\theta\pi$ is non degenerate and can be realized in the space of functions ${}^\theta W(g) = W(\theta(g))$, $W$ in the space of $\pi$. We take $A$ to be the operator on the space of $\pi$ which maps $W$ to ${}^\theta W$.

A $G$-module $\pi$ is called *unramified* if the space of $\pi$ contains a nonzero $K$-fixed vector. The dimension of the space of $K$-fixed vectors is bounded by one if $\pi$ is irreducible. If $\pi$ is $\theta$-invariant and unramified, and $v_0 \neq 0$ is a $K$-fixed vector in the space of $\pi$, then $Av_0$ is a multiple of $v_0$ (since $\theta K = K$), namely $Av_0 = cv_0$, with $c = \pm 1$. Replace $A$ by $cA$ to have $Av_0 = v_0$, and put $\pi(\theta) = A$.

When $\pi$ is (irreducible) unramified and has a Whittaker model, both normalizations of the intertwining operator are equal. In this case $\boldsymbol{\psi}$ is unramified (trivial on $R$ but not on $\boldsymbol{\pi}^{-1}R$, where $\boldsymbol{\pi}$ is a generator of the maximal ideal of $R$), and there exists a unique Whittaker function $W_0$ in the space of $\pi$ with respect to $\psi$ with $W_0 = 1$ on $K$. It is mapped by $\pi(\theta) = A$ to ${}^\theta W_0$, which satisfies ${}^\theta W_0(k) = 1$ for all $k$ in $K$ since $K$ is $\theta$-invariant. Namely $A$ maps the unique normalized (by $W_0(K) = 1$) $K$-fixed vector $W_0$ in the space of $\pi$ to the unique normalized $K$-fixed vector ${}^\theta W_0$ in the space of ${}^\theta\pi$, and we have ${}^\theta W_0 = W_0$.

For any (admissible) $\pi$ and (smooth) $f$ the convolution operator $\pi(fdg) = \int_G f(g)\pi(g)dg$ has finite rank. If $\pi$ is $\theta$-invariant put $\pi(fdg\times\theta) = \int_G f(g)\pi(g)\pi(\theta)dg$. Denote by tr $\pi(fdg\times\theta)$ the trace of the operator $\pi(fdg\times\theta)$. It depends on the choice of the Haar measure $dg$, but the (*twisted*) *character* $\chi_\pi$ of $\pi$ does not; $\chi_\pi$ is a locally-integrable complex-valued function on $G \times \theta$ (see [C], [H]) which is $\theta$-conjugacy invariant and locally-constant on the $\theta$-regular set, with tr $\pi(fdg \times \theta) = \int_G f(g)\chi_\pi(g \times \theta)dg$ for all $f$.

Local integrability is not used in this work; rather it is recovered for our twisted character.

## Small representation

To describe the $G$-module of interest in this paper, take $P$ to be the upper triangular parabolic subgroup of type (3,1), and fix its Levi factor to be $M = \{m = \mathrm{diag}(m_3, m_1);\ m_3 \in \mathrm{GL}(3,F),\ m_1 \in F^\times\}$. It is isomorphic to $\mathrm{GL}(3,F) \times F^\times$. Let $\delta$ denote (as above) the character $\delta(p) = |\,\mathrm{Ad}(p)|\,\mathrm{Lie}\,N|$ of $P$; it is trivial on the unipotent radical $N$ $(= F^3)$ of $P$. Then the value of $\delta$ at $p = mn$ is $|m_1^{-3}\det m_3|$. Denote by $I(\pi_1)$ the $G$-module $\pi = \mathrm{Ind}(\delta^{1/2}\pi_1; P, G)$ normalizedly induced from $\pi_1$ on $P$ to $G$. It is clear from [BZ] that when $\pi_1$ is self-contragredient and $I(\pi_1)$ is irreducible then $I(\pi_1)$ is $\theta$-invariant, and it is unramified if and only if $\pi_1$ is unramified.

*Our aim in this work is to compute the $\theta$-twisted character $\chi_{\pi_Y}$ of the* $\mathrm{GL}(4,F)$*-module* $\pi_Y = I_{(3,1)}(1_3 \times \chi_Y)$*, where* $1_3 \times \chi_Y$ *is the* $P$*-module* $\begin{pmatrix} m_3 & * \\ 0 & m_1 \end{pmatrix} \mapsto \chi_Y(m_1)$*,* $\chi_Y$ *is a quadratic character of* $F^\times$*,* $m_i \in \mathrm{GL}(i,F)$*, by purely local means.*

We begin by describing a useful model of our representation, in analogy with the models of [FK] and [FZ4] of analogous representations $I_{(2,1)}(1_2)$ of $\mathrm{PGL}(3,F)$ and $I_{(3,1)}(1_3)$ of $\mathrm{PGL}(4,F)$. Indeed we shall express $\pi_Y$ as an integral operator in a convenient model, and integrate the kernel over the diagonal to compute the character of $\pi_Y$.

Denote by $\mu = \mu_s$ the character $\mu(x) = |x|^{(s+1)/2}$ of $F^\times$, and by $\chi_Y$ a quadratic character of $F^\times$. This pair $(\mu, \chi_Y)$ defines a character $\mu_P = \mu_{s,Y,P}$ of $P$, trivial on $N$, by $\mu_P(p) = \mu((\det m_3)/{m_1}^3)\chi_Y(m_1)$ if $p = mn$ and $m = \begin{pmatrix} m_3 & 0 \\ 0 & m_1 \end{pmatrix}$ with $m_3$ in $\mathrm{GL}(3, F)$, $m_1$ in $\mathrm{GL}(1, F)$. If $s = 0$, then $\mu_P = \delta^{1/2}\chi_Y$, where viewed as a character on $P$, $\chi_Y$ takes the value $\chi_Y(m_1)$ at $p$. Let $W_s = W_s^Y$ be the space of complex-valued smooth functions $\psi$ on $G$ with $\psi(pg) = \mu_P(p)\psi(g)$ for all $p$ in $P$ and $g$ in $G$. The group $G$ acts on $W_s$ by right translation: $(\pi_s(g)\psi)(h) = \psi(hg)$. By definition, $I_{(3,1)}(1_3 \times \chi_Y)$ is the $G$-module $W_s$ with $s = 0$. The parameter $s$ is introduced for purposes of analytic continuation.

We prefer to work in another model $V_s = V_s^Y$ of the $G$-module $W_s$. Let $V$ denote the space of column 4-vectors over $F$. Let $V_s$ be the space of smooth complex-valued functions $\phi$ on $V - \{0\}$ with $\phi(\lambda \mathbf{v}) = \mu(\lambda)^{-4}\chi_Y(\lambda)\phi(\mathbf{v})$. The group $G$ acts on $V_s$ by $(\tau_s(g)\phi)(\mathbf{v}) = \mu(\det g)\phi({}^t g\mathbf{v})$. Let $\mathbf{v}_0 \neq 0$ be a vector of $V$ such that the line $\{\lambda\mathbf{v}_0; \lambda$ in $F\}$ is fixed under the action of ${}^tP$. Explicitly, we take $\mathbf{v}_0 = {}^t(0, 0, 0, 1)$. It is clear that the map $V_s \to W_s$, $\phi \mapsto \psi = \psi_\phi$, where $\psi(g) = (\tau_s(g)\phi)(\mathbf{v}_0) = \mu(\det g)\phi({}^t g\mathbf{v}_0)$, is a $G$-module isomorphism (check that $\psi_{\tau_s(g)\phi} = \pi_s(g)\psi_\phi$), with inverse $\psi \mapsto \phi = \phi_\psi$, $\phi(\mathbf{v}) = \mu(\det g)^{-1}\psi(g)$ if $\mathbf{v} = {}^t g\mathbf{v}_0$ ($G$ acts transitively on $V - \{0\}$).

For $\mathbf{v} = {}^t(x, y, z, t)$ in $V$ put $\|\mathbf{v}\| = \max(|x|, |y|, |z|, |t|)$. Let $V^0$ be the quotient of the set $V^1$ of $\mathbf{v}$ in $V$ with $\|\mathbf{v}\| = 1$, by the equivalence relation $\mathbf{v} \sim \alpha\mathbf{v}$ if $\alpha$ is a unit in $R$. Denote by $\mathbb{P}V$ the projective space of lines in $V - \{0\}$. If $\Phi$ is a function on $V - \{0\}$ with $\Phi(\lambda\mathbf{v}) = |\lambda|^{-4}\Phi(\mathbf{v})$ and $d\mathbf{v} = dx\ dy\ dz\ dt$, then $\Phi(\mathbf{v})d\mathbf{v}$ is homogeneous of degree zero. Define

$$\int_{\mathbb{P}V} \Phi(\mathbf{v})d\mathbf{v} \quad \text{to be} \quad \int_{V^0} \Phi(\mathbf{v})d\mathbf{v}.$$

Clearly we have

$$\int_{\mathbb{P}V} \Phi(\mathbf{v})d\mathbf{v} = \int_{\mathbb{P}V} \Phi(g\mathbf{v})d(g\mathbf{v}) = |\det g| \int_{\mathbb{P}V} \Phi(g\mathbf{v})d\mathbf{v}.$$

Put $\nu(x) = |x|$ and $m = 2(s - 1)$. Note that $\nu/\mu_s = \mu_{-s}$. Put $\langle \mathbf{w}, \mathbf{v}\rangle = {}^t\mathbf{w}J\mathbf{v}$. Then $\langle g\mathbf{w}, \theta(g)\mathbf{v}\rangle = \langle \mathbf{w}, \mathbf{v}\rangle$.

**1. Proposition.** *The operator*

$$T_s^Y : V_s \to V_{-s}, \qquad (T_s^Y\phi)(\mathbf{v}) = \int_{\mathbb{P}V} \phi(\mathbf{w})|\langle \mathbf{w}, \mathbf{v}\rangle|^m \chi_Y(\langle \mathbf{w}, \mathbf{v}\rangle)d\mathbf{w},$$

*converges on* $\mathrm{Re}\, s > 1/2$, *and satisfies there* $T_s^Y\tau_s(g) = \tau_{-s}(\theta(g))T_s^Y$ *for all* $g$ *in* $G$.

*Proof.* We have

$$\begin{aligned}
(T_s^Y(\tau_s(g)\phi))(\mathbf{v}) &= \textstyle\int (\tau_s(g)\phi)(\mathbf{w})|{}^t\mathbf{w}J\mathbf{v}|^m \chi_Y(\langle \mathbf{w}, \mathbf{v}\rangle)d\mathbf{w} \\
&= \mu(\det g) \textstyle\int \phi({}^t g\mathbf{w})|{}^t\mathbf{w}J\mathbf{v}|^m \chi_Y(\langle \mathbf{w}, \mathbf{v}\rangle)d\mathbf{w} \\
&= |\det g|^{-1}\mu(\det g) \textstyle\int \phi(\mathbf{w})|{}^t({}^t g^{-1}\mathbf{w})J\mathbf{v}|^m \chi_Y(\langle {}^t g^{-1}\mathbf{w}, \mathbf{v}\rangle)d\mathbf{w} \\
&= (\mu/\nu)(\det g) \textstyle\int \phi(\mathbf{w})|{}^t\mathbf{w}J \cdot J^{-1}g^{-1}J\mathbf{v}|^m \chi_Y(\langle \mathbf{w}, \theta({}^t g)\mathbf{v}\rangle)d\mathbf{w} \\
&= (\mu/\nu)(\det g) \textstyle\int \phi(\mathbf{w})|\langle \mathbf{w}, \theta({}^t g)\mathbf{v}\rangle|^m \chi_Y(\langle \mathbf{w}, \theta({}^t g)\mathbf{v}\rangle)d\mathbf{w} \\
&= (\nu/\mu)(\det\theta(g)) \cdot (T_s^Y\phi)({}^t\theta(g)\mathbf{v}) = [(\tau_{-s}(\theta(g)))(T_s^Y\phi)](\mathbf{v})
\end{aligned}$$

for the functional equation.

For the convergence, we may assume that $\phi = 1$ and ${}^t\mathbf{v} = (0,0,0,1)$, so that the integral is $\int_R |x|^m dx$, which converges for $\mathrm{Re}\, m > -1$. Our $m$ is $2s-2$, as required. □

The spaces $V_s$ are isomorphic to the space $W$ of locally-constant complex-valued functions $\phi$ on $V^1$ with $\phi(\lambda\mathbf{v}) = \chi_Y(\lambda)\phi(\mathbf{v})$ for all $\lambda \in R^\times$, and $T_s^Y$ is equivalent to an operator $T_s^{Y,0}$ on $W$. The proof of Proposition 1 implies also

**1. Corollary.** *The operator $T_s^{Y,0} \circ \tau_s(g^{-1})$ is an integral operator with kernel*

$$(\mu/\nu)(\det\theta(g))|\langle \mathbf{w}, \theta({}^t g^{-1})\mathbf{v}\rangle|^m \chi_Y(\langle \mathbf{w}, \theta({}^t g^{-1})\mathbf{v}\rangle) \qquad (\mathbf{v}, \mathbf{w} \textit{ in } V^1)$$

*and trace*

$$\mathrm{tr}[T_s^{Y,0} \circ \tau_s(g^{-1})] = (\nu/\mu)(\det g)\int_{V^0} |{}^t\mathbf{v}gJ\mathbf{v}|^m \chi_Y({}^t\mathbf{v}gJ\mathbf{v})d\mathbf{v}.$$

Next we normalize the operator $T^Y = T_s^{Y,0}$. Recalling that $\chi_Y$ is unramified ($=1$ on $R^\times$, $\chi_Y(\boldsymbol{\pi}) = -1$), we normalize $T^Y$ so that it acts trivially on the one-dimensional space of $K$-fixed vectors in $V_s$. This space is spanned by the function $\phi_0$ in $V_s$ with $\phi_0(\mathbf{v}) = 1$ for all $\mathbf{v}$ in $V^0$. This is the only case studied in full in this paper.

Denote again by $\boldsymbol{\pi}$ a generator of the maximal ideal of the ring $R$ of integers in our local nonarchimedean field $F$ of odd residual characteristic. Denote by $q$ the number of elements of the residue field $R/\boldsymbol{\pi}R$ of $R$. Normalize the absolute value by $|\boldsymbol{\pi}| = q^{-1}$, and the measures by $\mathrm{vol}\{|x| \le 1\} = 1$. Then $\mathrm{vol}\{|x| = 1\} = 1 - q^{-1}$, and the volume of $V^0$ is $(1-q^{-4})/(1-q^{-1}) = 1 + q^{-1} + q^{-2} + q^{-3}$.

**2. Proposition.** *If $\mathbf{v}_0 = {}^t(0,0,0,1)$, we have*

$$(T^Y\phi_0)(\mathbf{v}_0) = \frac{1+q^{-2(s+1)}}{1+q^{1-2s}}\phi_0(\mathbf{v}_0).$$

*When $s = 0$, the constant is $(1+q^{-2})(1+q)^{-1}$.*

*Proof.* Since $\chi_Y$ is unramified, we have

$$(T^Y\phi_0)(\mathbf{v}_0) = \int_{V^0} \phi_0(\mathbf{v})|{}^t\mathbf{v}J\mathbf{v}_0|^m \chi_Y({}^t\mathbf{v}J\mathbf{v}_0)d\mathbf{v} = \int_{V^0} |x|^m \chi_Y(x)dxdydzdt$$

$$= \left[\int_{||\mathbf{v}||\le 1} - \int_{||\mathbf{v}||<1}\right] |x|^m \chi_Y(x)dxdydzdt \Big/ \int_{|x|=1} dx$$

$$= (1+q^{-m-4})\int_{|x|\le 1} |x|^m \chi_Y(x)dx \Big/ \int_{|x|=1} dx = (1+q^{-2(s+1)})/(1+q^{1-2s}),$$

since $m = 2(s-1)$ and

$$\int_{|x|\le 1} |x|^m \chi_Y(x)dx = (1+q^{-m-1})^{-1}\int_{|x|=1} dx.$$

The proposition follows. □

Character computation for type I

For the $\theta$-conjugacy class of type I, represented by $g = t \cdot \mathrm{diag}(\mathbf{r}, \mathbf{s}, \mathbf{s}, \mathbf{r})$, the product

$$ {}^t\mathbf{v}gJ\mathbf{v} = (t, z, x, y) \begin{pmatrix} a_1\mathbf{r} & 0 & 0 & a_2D\mathbf{r} \\ 0 & b_1\mathbf{s} & b_2D\mathbf{s} & 0 \\ 0 & b_2\mathbf{s} & b_1\mathbf{s} & 0 \\ a_2\mathbf{r} & 0 & 0 & a_1\mathbf{r} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} t \\ z \\ x \\ y \end{pmatrix} $$

is equal to

$$ -t^2 a_2 D\mathbf{r} - z^2 b_2 D\mathbf{s} + x^2 b_2 \mathbf{s} + y^2 a_2 \mathbf{r}. $$

Note that $\mathbf{r}$ and $\mathbf{s}$ range over a set of representatives for $F^\times / N_{E/F} E^\times$.

By Corollary 1, we need to compute

$$ (\frac{\nu}{\mu})(\det g) \frac{\Delta(g\theta)}{\Delta_C(Ng)} \int_{V^0} |{}^t\mathbf{v}gJ\mathbf{v}|^m \chi_Y({}^t\mathbf{v}gJ\mathbf{v}) d\mathbf{v} $$

$$ = \frac{|\mathbf{rs}|^{1-s}|4a_2b_2D|}{|(a_1^2 - a_2^2 D)(b_1^2 - b_2^2 D)|^{s/2}} \int_{V^0} |\alpha|^{2(s-1)} \chi_Y(\alpha) dx dy dz dt. $$

Here $\alpha$ is $x^2 b_2 \mathbf{s} + y^2 a_2 \mathbf{r} - z^2 b_2 D\mathbf{s} - t^2 a_2 D\mathbf{r}$. Put $\mathbf{r}' = -\frac{a_2}{b_2}\frac{\mathbf{r}}{\mathbf{s}}$. Thus we need to compute the value at $s = 0$ of the product of

$$ \chi_Y(b_2\mathbf{s}) |\frac{\mathbf{r}}{\mathbf{s}}|^{-s} |4D\mathbf{r}'| |((\frac{a_1}{b_2})^2 - (\frac{a_2}{b_2})^2 D)((\frac{b_1}{b_2})^2 - D)|^{-s/2} $$

with the integral $I_s^Y(\mathbf{r}', D)$, where $Q = x^2 - \mathbf{r}y^2 - Dz^2 + \mathbf{r}Dt^2$ and

$$ I_s^Y(\mathbf{r}, D) = \int_{V^0} |Q|^{2(s-1)} \chi_Y(Q) dx dy dz dt. $$

**I. Theorem.** *When $Y/F$ is unramified, the value of $I_s^Y(\mathbf{r}, D)$ at $s = 0$ is 0.*

*Proof.* Consider the case when the quadratic form $Q = x^2 - \mathbf{r}y^2 - Dz^2 + \mathbf{r}Dt^2$ is anisotropic (does not represent zero). Thus $D = \boldsymbol{\pi}$ and $\mathbf{r} \in R^\times - R^{\times 2}$ (hence $|\mathbf{r}| = 1$, $|D| = 1/q$), or $D \in R^\times - R^{\times 2}$ and $\mathbf{r} = \boldsymbol{\pi}$. The second case being equivalent to the first, it suffices to deal with the first case.

The domain $\max\{|x|, |y|, |z|, |t|\} = 1$ is the disjoint union of $\{|x| = 1\}$, $\{|x| < 1, |y| = 1\}$, $\{|x| < 1, |y| < 1, |z| = 1\}$ and $\{|x| < 1, |y| < 1, |z| < 1, |t| = 1\}$. Note that $\chi_Y(x^2 - \mathbf{r}y^2 - Dz^2 + \mathbf{r}Dt^2)$ is equal to 1 on the first two subdomains and equals $-1$ on the other two. Thus the integral $I_s^Y(\mathbf{r}, D)$ is the quotient by $\int_{|x|=1} dx$ of

$$ \int_{|x|=1} dx + \int \int_{|x|<1, |y|=1} dx dy - q^{-m} \int \int \int_{|x|<1, |y|<1, |z|=1} dx dy dz $$

$$-q^{-m}\int\int\int\int_{|x|<1,|y|<1,|z|<1,|t|=1} dxdydzdt$$

$$= 1+q^{-1}-q^{-m-2}-q^{-m-3} = 1+q^{-1}-q^{-2s}-q^{-2s-1}.$$

The value at $s=0$ is 0 and thus the theorem follows when the quadratic form is anisotropic.

We then turn to the case when the quadratic form is isotropic. Recall that $\mathbf{r}$ ranges over a set of representatives for $F^\times/N_{E/F}E^\times$, $E=F(\sqrt{D})$. Thus $D\in F-F^2$, and we may assume that $|D|$ and $|\mathbf{r}|$ lie in $\{1,q^{-1}\}$.

**I.1. Proposition.** *When the quadratic form $x^2-\mathbf{r}y^2-Dz^2+\mathbf{r}Dt^2$ is isotropic, $\mathbf{r}$ lies in $N_{E/F}E^\times$, and we may assume that the quadratic form takes one of three shapes:*

$$x^2-y^2-Dz^2+Dt^2,\ \ D\in R^\times-R^{\times 2};\quad x^2+\boldsymbol{\pi}y^2-\boldsymbol{\pi}z^2-\boldsymbol{\pi}^2t^2;\quad x^2-y^2-\boldsymbol{\pi}z^2+\boldsymbol{\pi}t^2.$$

*Proof.* (1) If $E/F$ is unramified, then $|D|=1$, thus $D\in R^\times-R^{\times 2}$. The norm group $N_{E/F}E^\times$ is $\pi^{2\mathbb{Z}}R^\times$. If $x^2-\mathbf{r}y^2-Dz^2+\mathbf{r}Dt^2$ represents 0 then $\mathbf{r}\in R^\times$, so we may take $\mathbf{r}=1$.
(2) If $E/F$ is ramified then $|D|=q^{-1}$ and $N_{E/F}E^\times=(-D)^{\mathbb{Z}}R^{\times 2}$. The form $x^2-\mathbf{r}y^2-Dz^2+\mathbf{r}Dt^2$ represents zero when $\mathbf{r}\in R^{\times 2}$ or $\mathbf{r}\in -DR^{\times 2}$. Then the form can be taken to be $x^2+Dy^2-Dz^2-D^2t^2$ with $\mathbf{r}=-D$ and $|D\mathbf{r}|=q^{-2}$, or $x^2-y^2-Dz^2+Dt^2$ with $\mathbf{r}=1$ and $|D\mathbf{r}|=q^{-1}$. The proposition follows. □

The set $V^0=V/\sim$, where $V=\{\mathbf{v}=(x,y,z,t)\in R^4;\max\{|x|,|y|,|z|,|t|\}=1\}$ and $\sim$ is the equivalence relation $\mathbf{v}\sim\alpha\mathbf{v}$ for $\alpha\in R^\times$, is the disjoint union of the subsets

$$V_n^0=V_n^0(\mathbf{r},D)=V_n(\mathbf{r},D)/\sim,$$

where

$$V_n=V_n(\mathbf{r},D)=\{\mathbf{v};\max\{|x|,|y|,|z|,|t|\}=1,|x^2-\mathbf{r}y^2-Dz^2+\mathbf{r}Dt^2|=1/q^n\},$$

over $n\geq 0$, and of $\{\mathbf{v};x^2-\mathbf{r}y^2-Dz^2+\mathbf{r}Dt^2=0\}/\sim$, a set of measure zero.
Thus the integral $I_s^Y(\mathbf{r},D)$ coincides with the sum

$$\sum_{n=0}^{\infty}(-1)^n q^{-nm}\operatorname{vol}(V_n^0(\mathbf{r},D)).$$

When the quadratic form represents zero the problem is then to compute the volumes

$$\operatorname{vol}(V_n^0(\mathbf{r},D))=\operatorname{vol}(V_n(\mathbf{r},D))/(1-1/q)\qquad(n\geq 0).$$

We need the following Technical Lemma.

**I.0. Lemma.** *When* $c^2 \in R^{\times 2}$ *and* $n \geq 1$, *we have*

$$\int_{|c^2-x^2|=q^{-n}} dx \;=\; \frac{2}{q^n}\left(1-\frac{1}{q}\right).$$

*Proof.* Making the change of variables $y = cx$, we have

$$\int_{|c^2-x^2|=q^{-n}} \dots dx \;=\; \int_{|1-y^2|=q^{-n}} \dots dy.$$

From $|1-y^2| = |1-y||1+y| = q^{-n}$ it follows that

$$y = \pm 1 + \varepsilon \boldsymbol{\pi}^n;\ |\varepsilon| = 1,\ dy = \frac{d\varepsilon}{q^n}.$$

Thus the integral is equal to

$$\int_{y=\pm 1+\varepsilon\boldsymbol{\pi}^n,|\varepsilon|=1} dy \;=\; \frac{2}{q^n}\int_{|\varepsilon|=1} d\varepsilon \;=\; \frac{2}{q^n}\left(1-\frac{1}{q}\right).$$

The lemma follows. □

**I.1. Lemma.** *When* $D = \boldsymbol{\pi}$ *and* $\mathbf{r} = 1$, *thus* $|\mathbf{r}D| = 1/q$, *we have*

$$\mathrm{vol}(V_n^0) = \begin{cases} 1-1/q, & \textit{if } n = 0, \\ q^{-1}(1-1/q)(2+1/q), & \textit{if } n = 1, \\ 2q^{-n}(1-1/q)(1+1/q), & \textit{if } n \geq 2. \end{cases}$$

*Proof.* In our case

$$V_0 = V_0(1,\boldsymbol{\pi}) = \{(x,y,z,t);\, \max\{|x|,|y|,|z|,|t|\} = 1,\, |x^2-y^2-\boldsymbol{\pi}z^2+\boldsymbol{\pi}t^2| = 1\}.$$

Since $|z| \leq 1$ and $|t| \leq 1$, we have $|\boldsymbol{\pi}(z^2-t^2)| < 1$, and

$$1 = |x^2-y^2-\boldsymbol{\pi}z^2+\boldsymbol{\pi}t^2| = |x^2-y^2| = |x-y||x+y|.$$

Thus $|x-y| = |x+y| = 1$, and if $|x| \neq |y|$, $|x \pm y| = \max\{|x|,|y|\}$. We split $V_0$ into three distinct subsets, corresponding to the cases $|x| = |y| = 1$; $|x| = 1$, $|y| < 1$; and $|x| < 1$, $|y| = 1$. The volume is then

$$\mathrm{vol}(V_0) = \int_{|t|\leq 1}\int_{|z|\leq 1}\int_{|x|=1}\left[\int_{|y|=1,|x-y|=|x+y|=1}\right] dy\,dx\,dz\,dt$$

$$+\int_{|t|\le 1}\int_{|z|\le 1}\left[\int_{|x|=1}\int_{|y|<1}+\int_{|x|<1}\int_{|y|=1}\right]dydxdzdt$$

$$=\int_{|x|=1}\left[\int_{|y|=1,|x-y|=|x+y|=1}\right]dydx+\frac{2}{q}\left(1-\frac{1}{q}\right)=\left(1-\frac{1}{q}\right)^2.$$

*Consider the case of* $V_n$ with $n\ge 2$. If $|x|=1$, then put $c=c(x,t,z)=x^2+\boldsymbol{\pi}(t^2-z^2)$. Since $|\boldsymbol{\pi}(t^2-z^2)|<1$, we have $c(x,t,z)\in R^{\times 2}$, and we can apply Lemma I.0. Thus we obtain

$$\int_{|t|\le 1}\int_{|z|\le 1}\int_{|x|=1}\int_{|c-y^2|=q^{-n}}dydxdzdt=\left(1-\frac{1}{q}\right)\frac{2}{q^n}\left(1-\frac{1}{q}\right)=\frac{2}{q^n}\left(1-\frac{1}{q}\right)^2.$$

If $|x|<1$ it follows that $|y|<1$. Since $\max\{|x|,|y|,|z|,|t|\}=1$ and $n\ge 2$ it follows that $|t|=|z|=1$. Indeed, if, say, $|t|=1$ but $|z|<1$, then $|x^2-y^2-\boldsymbol{\pi}z^2+\boldsymbol{\pi}t^2|=|\boldsymbol{\pi}t^2|=1/q$, which is a contradiction. Further, dividing by $\boldsymbol{\pi}$ we obtain $|z^2+(y^2-x^2)/\boldsymbol{\pi}-t^2|=q^{1-n}$. Put $c=c(x,y,z)=z^2+(y^2-x^2)/\boldsymbol{\pi}$. Since $|(y^2-x^2)/\boldsymbol{\pi}|<1$, we have $c\in R^{\times 2}$, and using Lemma I.0 we obtain

$$\int_{|x|<1}\int_{|y|<1}\int_{|z|=1}\int_{|c-t^2|=q^{1-n}}dtdzdydx=\frac{1}{q^2}\left(1-\frac{1}{q}\right)\frac{2}{q^{n-1}}\left(1-\frac{1}{q}\right)=\frac{2}{q^{n+1}}\left(1-\frac{1}{q}\right)^2.$$

Adding the two cases we have ($n\ge 2$)

$$\mathrm{vol}(V_n)=\frac{2}{q^n}\left(1-\frac{1}{q}\right)^2+\frac{2}{q^{n+1}}\left(1-\frac{1}{q}\right)=\frac{2}{q^n}\left(1+\frac{1}{q}\right)\left(1-\frac{1}{q}\right)^2.$$

*Consider the case* $n=1$. The case $|x|=1$, is exactly the same as for $n\ge 2$. The contribution is $2/q(1-1/q)^2$. Now if $|x|<1$ then $|y|<1$ and $\max\{|z|,|t|\}=1$. We have $|x^2-y^2-\boldsymbol{\pi}z^2+\boldsymbol{\pi}t^2|=|\boldsymbol{\pi}(z^2-t^2)|=q^{-1}$. Dividing by $\boldsymbol{\pi}$ gives $|z^2-t^2|=1$. The volume of this subset is

$$\frac{1}{q^2}\left[\int_{|z|=1}\int_{|z^2-t^2|=1}dtdz+\int_{|z|<1}\int_{|t|=1}dtdz\right]$$

$$=\frac{1}{q^2}\left[\left(1-\frac{1}{q}\right)\left(1-\frac{2}{q}\right)+\frac{1}{q}\left(1-\frac{1}{q}\right)\right]=\frac{1}{q^2}\left(1-\frac{1}{q}\right)^2.$$

Adding the two cases, we have

$$\mathrm{vol}(V_1)=\frac{2}{q}\left(1-\frac{1}{q}\right)^2+\frac{1}{q^2}\left(1-\frac{1}{q}\right)^2=\frac{1}{q}\left(2+\frac{1}{q}\right)\left(1-\frac{1}{q}\right)^2.$$

The lemma follows. □

**I.2. Lemma.** *When $D = \boldsymbol{\pi}$ and $\mathbf{r} = -\boldsymbol{\pi}$, thus $|\mathbf{r}D| = 1/q^2$, we have*

$$\mathrm{vol}(V_n^0) = \begin{cases} 1, & \text{if } n = 0, \\ q^{-1}(1-1/q), & \text{if } n = 1, \\ q^{-2}(2-1/q-2/q^2), & \text{if } n = 2, \\ 2q^{-n}(1-1/q)(1+1/q), & \text{if } n \geq 3. \end{cases}$$

*Proof. To compute* $\mathrm{vol}(V_0)$, recall that in our case

$$V_0 = \{(x,y,z,t); \max\{|x|,|y|,|z|,|t|\} = 1, |x^2 + \boldsymbol{\pi}(y^2 - z^2) - \boldsymbol{\pi}^2 t^2| = 1\}.$$

Since $|y| \leq 1$, $|z| \leq 1$, $|t| \leq 1$, we have $|x^2 + \boldsymbol{\pi}(y^2 - z^2) - \boldsymbol{\pi}^2 t^2| = |x^2| = 1$, and so

$$\mathrm{vol}(V_0) = \int_{|t|\leq 1}\int_{|z|\leq 1}\int_{|y|\leq 1}\int_{|x|=1} dxdydzdt = 1 - \frac{1}{q}.$$

*To compute* $\mathrm{vol}(V_n)$, $n \geq 1$, recall that

$$V_n = \{(x,y,z,t); \max\{|x|,|y|,|z|,|t|\} = 1, |x^2 + \boldsymbol{\pi}(y^2 - z^2) - \boldsymbol{\pi}^2 t^2| = 1/q^n\}.$$

Assume that $|x| = 1$. Then

$$1 = |x^2| = |x^2 + \boldsymbol{\pi}(y^2 - z^2) - \boldsymbol{\pi}^2 t^2| = 1/q^n < 1.$$

Thus we have that $|x| < 1$ and $\max\{|y|,|z|,|t|\} = 1$.

*Consider the case* $n = 1$. Then $|y^2 - z^2| = |y \pm z| = 1$. We have

$$\mathrm{vol}(V_1) = \int_{|t|\leq 1}\int_{|x|<1}\left[\int_{|y|=1}\int_{|y\pm z|=1} + \int_{|y|<1}\int_{|z|=1}\right] dzdydxdt$$
$$= \frac{1}{q}\left[\left(1-\frac{1}{q}\right)\left(1-\frac{2}{q}\right) + \frac{1}{q}\left(1-\frac{1}{q}\right)\right] = \frac{1}{q}\left(1-\frac{1}{q}\right)^2.$$

*Consider the case* $n \geq 3$. As in the analogous case of Lemma I.1, we consider the cases of $|y| = 1$ and $|y| < 1$. Adding the two cases we have ($n \geq 3$)

$$\mathrm{vol}(V_n) = \frac{2}{q^n}\left(1-\frac{1}{q}\right)^2 + \frac{2}{q^{n+1}}\left(1-\frac{1}{q}\right) = \frac{2}{q^n}\left(1+\frac{1}{q}\right)\left(1-\frac{1}{q}\right)^2.$$

*Consider the case* $n = 2$. If $|y| = 1$ we apply Lemma I.0 (as in the case $n \geq 3$) and the contribution is $2/q^2(1-1/q)^2$. Now if $|y| < 1$ then $|z| < 1$ and $|t| = 1$. The contribution from this subset is

$$\int_{|y|<1}\int_{|z|<1}\int_{|t|=1}\int_{|t^2-(x/\pi)^2|=1} dxdtdzdy = \frac{1}{q^2}\left(1-\frac{1}{q}\right)\frac{1}{q}\left(1-\frac{2}{q}\right).$$

Adding the two cases (see Lemma I.1 for details), we obtain

$$\mathrm{vol}(V_2) = \frac{2}{q^2}\left(1-\frac{1}{q}\right)^2 + \frac{1}{q^3}\left(1-\frac{1}{q}\right)\left(1-\frac{2}{q}\right) = \frac{1}{q^2}\left(1-\frac{1}{q}\right)\left(2-\frac{1}{q}-\frac{2}{q^2}\right).$$

The lemma follows. $\square$

**I.3. Lemma.** *When $E/F$ is unramified, thus $|\mathbf{r}D| = 1$, we have*

$$\mathrm{vol}(V_n^0) = \begin{cases} 1 - 1/q^2, & \text{if } n = 0, \\ q^{-n}(1-1/q)(1+2/q+1/q^2), & \text{if } n \geq 1. \end{cases}$$

*Proof. First we compute* $\mathrm{vol}(V_0)$. Recall that in our case

$$V_0 = \{(x,y,z,t); \max\{|x|,|y|,|z|,|t|\} = 1, |x^2 - y^2 - D(z^2 - t^2)| = 1\}.$$

Since $|x^2 - y^2 - D(z^2 - t^2)| \leq \max\{|x|,|y|,|z|,|t|\}$,

$$V_0 = \{(x,y,z,t) \in R^4; |x^2 - y^2 - D(z^2 - t^2)| = 1\}.$$

Make the change of variables $x' = x + y$, $y' = x - y$, $z' = z + t$, $t' = z - t$. Renaming $x'$, $y'$, $z'$, $t'$ as $x$, $y$, $z$, $t$, we obtain

$$V_0 = \{(x,y,z,t) \in R^4; |xy - Dzt| = 1\}.$$

Assume that $|xy| < 1$. Since $|xy - Dzt| = 1$, it follows that $|zt| = |z| = |t| = 1$. The contribution from the set $|xy| < 1$ is

$$\int_{|t|=1} \int_{|z|=1} \left[ \int_{|x|<1} \int_{|y|\leq 1} + \int_{|x|=1} \int_{|y|<1} \right] dydxdzdt$$

$$= \left(1 - \frac{1}{q}\right)^2 \left(\frac{1}{q} + \left(1 - \frac{1}{q}\right)\frac{1}{q}\right) = \frac{1}{q}\left(1 - \frac{1}{q}\right)^2 \left(2 - \frac{1}{q}\right).$$

Note that the contribution from $|xy| = 1$, $|zt| < 1$, is the same and equals

$$\frac{1}{q}\left(1 - \frac{1}{q}\right)^2 \left(2 - \frac{1}{q}\right).$$

We are left with the case $|xy| = |zt| = 1$, i.e. $|x| = |y| = |z| = |t| = 1$. If $|x| = |y| = |z| = 1$ we introduce $U(x,y,z) = \{t;\ |t| = 1,\ |xy - Dzt| = 1\}$, a set of volume $1 - 2/q$. The contribution from this case is

$$\int_{|x|=1} \int_{|y|=1} \int_{|z|=1} \int_{U(x,y,z)} dtdzdydx = \left(1 - \frac{1}{q}\right)^3 \left(1 - \frac{2}{q}\right).$$

Thus we obtain

$$\mathrm{vol}(V_0) = \frac{2}{q}\left(1 - \frac{1}{q}\right)^2 \left(2 - \frac{1}{q}\right) + \left(1 - \frac{1}{q}\right)^3 \left(1 - \frac{2}{q}\right) = \left(1 - \frac{1}{q}\right)^2 \left(1 + \frac{1}{q}\right).$$

*Next we compute* $\mathrm{vol}(V_n)$, $n \geq 1$. Recall that in our case

$$V_n = \{(x,y,z,t); \max\{|x|,|y|,|z|,|t|\} = 1, |x^2 - y^2 - D(z^2 - t^2)| = 1/q^n\}.$$

Making the change of variables $u = x + y$, $v = x - y$, we obtain

$$V_n = \{(u,v,z,t); \max\{|u+v|,|u-v|,|z|,|t|\} = 1, |uv - D(z^2 - t^2)| = 1/q^n\}.$$

Since the set $\{v = 0\}$ is of measure zero, we assume that $v \neq 0$. Then $|uv - D(z^2 - t^2)| = 1/q^n$ implies that $u = D(z^2 - t^2)v^{-1} + wv^{-1}\boldsymbol{\pi}^n$, where $|w| = 1$. There are two cases.

Assume that $|v| = 1$. Note that if $|z^2 - t^2| = 1$, then $\max\{|z|,|t|\} = 1$, and if $|z^2 - t^2| < 1$, then (since $n \geq 1$)

$$|u| = |D(z^2 - t^2)v^{-1} + wv^{-1}\boldsymbol{\pi}^n| \leq \max\{|z^2 - t^2|, q^{-n}\} < 1,$$

and consequently $|u + v| = |v| = 1$. So $|v| = 1$ implies that $\max\{|u+v|,|u-v|,|z|,|t|\} = 1$. Further, since $|v| = 1$, we have $du = q^{-n}dw$. Thus the contribution from the set with $|v| = 1$ is

$$\int_{|t|\leq 1}\int_{|z|\leq 1}\int_{|v|=1}\int_{|uv-D(z^2-t^2)|=1/q^n} dudvdzdt = \int_{|v|=1}\int_{|w|=1}\frac{dw}{q^n}dv = \frac{1}{q^n}\left(1 - \frac{1}{q}\right)^2.$$

Assume that $|v| < 1$ and $|u| = 1$. Thus $\max\{|u+v|,|u-v|,|z|,|t|\} = 1$. We write $v = D(z^2 - t^2)u^{-1} + wu^{-1}\boldsymbol{\pi}^n$ where $|w| = 1$ and $dv = q^{-n}dw$. Since

$$|z^2 - t^2| \leq \max\{|v|, q^{-n}\} < 1,$$

it follows that $|z^2 - t^2| < 1$. Note that

$$\int\int_{|z^2-t^2|<1} dzdt = \int_{|z|=1}\int_{|z^2-t^2|<1} dtdz + \int_{|z|<1}\int_{|t|<1} dt = \left(1 - \frac{1}{q}\right)\frac{2}{q} + \frac{1}{q^2} = \frac{1}{q}\left(2 - \frac{1}{q}\right).$$

The volume of this subset equals

$$\int\int_{|z^2-t^2|<1}\int_{|u|=1}\int_{|uv-D(z^2-t^2)|=1/q^n} dvdudzdt = \frac{1}{q}\left(2 - \frac{1}{q}\right)\frac{1}{q^n}\int_{|u|=1}\int_{|w|=1} dwdu$$

$$= \frac{1}{q}\frac{1}{q^n}\left(1 - \frac{1}{q}\right)^2\left(2 - \frac{1}{q}\right).$$

Assume that $|v| < 1$ and $|u| < 1$. Then we have $|u \pm v| < 1$ and thus $\max\{|z|,|t|\} = 1$. Since $|uv - D(z^2 - t^2)| < 1$ it follows that $|z^2 - t^2| < 1$. So we have $|z| = |t| = 1$. Put $c = c(z,u,v) = z^2 - uvD^{-1}$. Then $c \in R^{\times 2}$ (since $|uvD^{-1}| < 1$). Dividing by $D$, we have

$$\frac{1}{q^n} = |uv - D(z^2 - t^2)| = |c - t^2|.$$

Applying Lemma I.0, the contribution from this subset is equal to

$$\int_{|u|<1}\int_{|v|<1}\int_{|z|=1}\int_{|c-t^2|=q^{-n}} dtdzdvdu = \frac{1}{q^2}\frac{2}{q^n}\left(1-\frac{1}{q}\right)^2.$$

Adding the contributions from $|v|=1$ and $|v|<1$ we obtain

$$\begin{aligned}\operatorname{vol}(V_n) &= \frac{1}{q^n}\left(1-\frac{1}{q}\right)^2 + \frac{1}{q}\frac{1}{q^n}\left(1-\frac{1}{q}\right)^2\left(2-\frac{1}{q}\right) + \frac{1}{q^2}\frac{2}{q^n}\left(1-\frac{1}{q}\right)^2 \\ &= \frac{1}{q^n}\left(1-\frac{1}{q}\right)^2\left(1+\frac{2}{q}+\frac{1}{q^2}\right).\end{aligned}$$

The lemma follows. □

*Proof of Theorem I.* We are now ready to complete the proof of Theorem I in the isotropic case. Recall that we need to compute the value at $s=0$ ($m=-2$) of $I_s^Y(\mathbf{r},D)$. Here $I_s^Y(\mathbf{r},D)$ coincides with the sum

$$\sum_{n=0}^{\infty}(-1)^n q^{-nm}\operatorname{vol}(V_n^0(\mathbf{r},D))$$

which converges for $m>-1$ by Proposition 1 or alternatively by Lemmas I.1-I.3. The value at $m=-2$ is obtained then by analytic continuation of this sum.
*Case of Lemma* I.1. The integral $I_s^Y(\mathbf{r},D)$ is equal to

$$\operatorname{vol}(V_0^0) - q^{-m}\operatorname{vol}(V_1^0) + \sum_{n=2}^{\infty}(-1)^n q^{-nm}\operatorname{vol}(V_n^0)$$

$$= 1-\frac{1}{q}-\frac{1}{q}\left(1-\frac{1}{q}\right)\left(2+\frac{1}{q}\right)\frac{1}{q^m} + 2\left(1-\frac{1}{q}\right)\left(1+\frac{1}{q}\right)q^{-2(m+1)}\left(1+\frac{1}{q^{m+1}}\right)^{-1}.$$

When $m=-2$, this is

$$1-\frac{1}{q}-q\left(2-\frac{1}{q}-\frac{1}{q^2}\right)+2\left(1-\frac{1}{q}\right)\left(1+\frac{1}{q}\right)\frac{q^2}{1+q}=0.$$

*Case of Lemma* I.2. The integral $I_s^Y(\mathbf{r},D)$ is equal to

$$\operatorname{vol}(V_0^0) - q^{-m}\operatorname{vol}(V_1^0) + q^{-2m}\operatorname{vol}(V_2^0) + \sum_{n=3}^{\infty}(-1)^n q^{-nm}\operatorname{vol}(V_n^0)$$

$$= 1-\frac{1}{q}\left(1-\frac{1}{q}\right)q^{-m} + \frac{1}{q^2}\left(2-\frac{1}{q}-\frac{2}{q^2}\right)q^{-2m}$$

$$-2\left(1-\frac{1}{q}\right)\left(1+\frac{1}{q}\right)q^{-3(m+1)}\left(1+\frac{1}{q^{m+1}}\right)^{-1}.$$

When $m=-2$, this is

$$1-\frac{1}{q}\left(1-\frac{1}{q}\right)q^2+\frac{1}{q^2}\left(2-\frac{1}{q}-\frac{2}{q^2}\right)q^4-2\left(1-\frac{1}{q}\right)\left(1+\frac{1}{q}\right)\frac{q^3}{1+q}.$$

Once simplified this is equal to 0.
*Case of Lemma* I.3. The integral $I_s^Y(\mathbf{r},D)$ is equal to

$$\mathrm{vol}(V_0^0)+\sum_{n=1}^{\infty}(-1)^n q^{-nm}\,\mathrm{vol}(V_n^0)$$

$$=1-\frac{1}{q^2}-\left(1-\frac{1}{q}\right)\left(1+\frac{2}{q}+\frac{1}{q^2}\right)q^{-(m+1)}\left(1+\frac{1}{q^{m+1}}\right)^{-1}.$$

When $m=-2$, this is

$$=1-\frac{1}{q^2}-\left(1-\frac{1}{q}\right)\left(1+\frac{1}{q}\right)^2\frac{q}{1+q}=0.$$

The theorem follows. □

## Character computation for type II

For the $\theta$-conjugacy class of type II, represented by $g=t\cdot\mathrm{diag}(\mathbf{r},\mathbf{s},\mathbf{s},\mathbf{r})$, the product

$${}^t\mathbf{v}gJ\mathbf{v}=(t,z,x,y)\begin{pmatrix}a_1\mathbf{r}&0&0&a_2D\mathbf{r}\\0&b_1\mathbf{s}&b_2AD\mathbf{s}&0\\0&b_2\mathbf{s}&b_1\mathbf{s}&0\\a_2\mathbf{r}&0&0&a_1\mathbf{r}\end{pmatrix}\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&-1&0&0\\-1&0&0&0\end{pmatrix}\begin{pmatrix}t\\z\\x\\y\end{pmatrix}$$

is equal to

$$-t^2a_2D\mathbf{r}-z^2b_2AD\mathbf{s}+x^2b_2\mathbf{s}+y^2a_2\mathbf{r}=b_2\mathbf{s}(x^2-y^2\mathbf{r}'-z^2AD+t^2D\mathbf{r}').$$

Here $a_1+a_2\sqrt{D}\in E_1^\times$ ($E_1=F(\sqrt{D})$) and $b_1+b_2\sqrt{AD}\in E_2^\times$ ($E_2=F(\sqrt{AD})$), and $\mathbf{r}'=-a_2\mathbf{r}/b_2\mathbf{s}$. As $\mathbf{r}$ ranges over a set of representatives for $F^\times/N_{E_1/F}E_1^\times$ (and $\mathbf{s}$ for $F^\times/N_{E_2/F}E_2^\times$), we may rename $\mathbf{r}'$ by $\mathbf{r}$.

Thus, by Corollary 1, we need to compute the value at $s=0$ of the product of

$$\chi_Y(b_2\mathbf{s})|\frac{\mathbf{r}}{\mathbf{s}}|^{-s}|4\mathbf{r}'D\sqrt{A}||((\frac{a_1}{b_2})^2-(\frac{a_2}{b_2})^2D)((\frac{b_1}{b_2})^2-AD)|^{-s/2}$$

and the value when $\mathbf{r}$ is $\mathbf{r}'$ and $Q = x^2 - \mathbf{r}y^2 - ADz^2 + \mathbf{r}Dt^2$ of the integral

$$I_s^Y(\mathbf{r}, A, D) = \int_{V^0} |Q|^{2(s-1)} \chi_Y(Q) dx dy dz dt.$$

The property of the numbers $A$, $D$ and $AD$ that we need is that their square roots generate the three distinct quadratic extensions of $F$. Thus we may assume that $\{A, D, AD\} = \{u, \boldsymbol{\pi}, u\boldsymbol{\pi}\}$, where $u \in R^\times - R^{\times 2}$. Of course with this normalization $AD$ is no longer the product of $A$ and $D$, but its representative in the set $\{1, u, \boldsymbol{\pi}, u\boldsymbol{\pi}\}$ mod $F^{\times 2}$. Since $\mathbf{r}$ ranges over a set of representatives for $F^\times / N_{E_1/F} E_1^\times$, it can be assumed to range over $\{1, \boldsymbol{\pi}\}$ if $D = u$, and over $\{1, u\}$ if $|D| = |\boldsymbol{\pi}|$.

In this section we prove

**II. Theorem.** *When $Y/F$ is unramified, the value of*

$$\chi_Y(b_2\mathbf{s}) |4\mathbf{r}D\sqrt{A}| I_s^Y(\mathbf{r}, A, D) / (T^Y \phi_0)(\mathbf{v}_0)$$

*at $s = 0$ is $-2\chi_Y(b_2\mathbf{s})\delta(Y, E_3)$.*

Recall that $E_3 = F(\sqrt{A})$. As usual, $\delta(Y, E_3)$ is 1 if $Y = E_3$ and 0 if $Y \neq E_3$.

The meaning of this result is that the twisted character of $\pi_Y$ on elements of tori of type II relates to values of the trivial character on $\mathbf{C}_Y(F)$, $Y = E_3$, on the torus which splits over $E$. It does not relate to such values on $\mathbf{C}_{Y'}(F)$, $Y' \neq E_3$.

To prove this theorem we need some lemmas.

Recall Theorem 6.26 and Theorem 6.27 of the book "Finite Fields" [LN] by Lidl and Niederreiter:

**Lemma (FF).** *Let $f(x_1, x_2, ..., x_n)$ be a quadratic form in $n$ variables. Let $\eta$ be the quadratic character of $\mathbb{F}_q$ : its value on $\mathbb{F}_q^{\times 2}$ is 1, on $\mathbb{F}_q^\times - \mathbb{F}_q^{\times 2}$ its value is $-1$, and $\eta(0) = 0$. The number of solutions in $\mathbb{F}_q$ of the quadratic equation $f(x_1, x_2, ..., x_n) = b$ ($b \in F_q$) is as follows.*

*(i) If $n$ is an odd integer, the number is*

$$q^{n-1} + q^{(n-1)/2} \eta((-1)^{(n-1)/2} b \det(f)).$$

*(ii) If $n$ is even, then putting $v(b) = -1$ if $b \neq 0$, and $v(0) = q - 1$, the number is*

$$q^{n-1} + v(b) q^{(n-2)/2} \eta((-1)^{n/2} \det(f)).$$

*Here* $\det(f)$ *is the determinant of the symmetric matrix representing the quadratic form $f(x_1, x_2, ..., x_n)$.*

**II.0. Lemma.** *The quadratic form* $x^2 - \mathbf{r}y^2 - ADz^2 + \mathbf{r}Dt^2$ *takes one of six forms*: $x^2 - y^2 + \boldsymbol{\pi}(t^2 - uz^2)$, $x^2 - uy^2 + u\boldsymbol{\pi}(t^2 - z^2)$, $x^2 - y^2 + ut^2 - u\boldsymbol{\pi}z^2$, $x^2 - y^2 - uz^2 + \boldsymbol{\pi}t^2$, $x^2 - \boldsymbol{\pi}y^2 + u\boldsymbol{\pi}(t^2 - z^2)$, $x^2 - uy^2 - uz^2 + u\boldsymbol{\pi}t^2$, *where* $u \in R^\times - R^{\times 2}$. *It is always isotropic.*

*Proof.* (1) If $E_1/F$ is unramified then $D = u$ where $u \in R^\times - R^{\times 2}$. The norm group $N_{E_1/F}E_1^\times$ is $\boldsymbol{\pi}^{2\mathbb{Z}}R^\times$. So $\mathbf{r} = 1$ or $\boldsymbol{\pi}$ and $A = \boldsymbol{\pi}$. We obtain two quadratic forms: $x^2 - y^2 - u\boldsymbol{\pi}z^2 + ut^2$, and $x^2 - \boldsymbol{\pi}y^2 - u\boldsymbol{\pi}(z^2 - t^2)$.
(2) If $E_1/F$ is ramified then $D = \boldsymbol{\pi}$ and $N_{E_1/F}E_1^\times = (-D)^{\mathbb{Z}}R^{\times 2}$. Then $\mathbf{r} = 1$ or $u$, and $A = u$ or $u\boldsymbol{\pi}$. Note that if $A = u\boldsymbol{\pi}$ we take $AD = u$. We obtain the following quadratic forms: if $\mathbf{r} = 1$, $A = u$ we have $x^2 - y^2 - \boldsymbol{\pi}(uz^2 - t^2)$; if $\mathbf{r} = 1$, $A = u\boldsymbol{\pi}$ we have $x^2 - y^2 - uz^2 + \boldsymbol{\pi}t^2$; if $\mathbf{r} = u$, $A = u$ we have $x^2 - uy^2 - u\boldsymbol{\pi}(z^2 - t^2)$; if $\mathbf{r} = u$, $A = u\boldsymbol{\pi}$ we have $x^2 - uy^2 - uz^2 + u\boldsymbol{\pi}t^2$. The proposition follows. □

The set $V^0 = V/\sim$, where $V = \{\mathbf{v} = (x, y, z, t) \in R^4; \max\{|x|, |y|, |z|, |t|\} = 1\}$ and $\sim$ is the equivalence relation $\mathbf{v} \sim \alpha\mathbf{v}$ for $\alpha \in R^\times$, is the disjoint union of the subsets

$$V_n^0 = V_n^0(\mathbf{r}, A, D) = V_n(\mathbf{r}, A, D)/\sim,$$

where

$$V_n = V_n(\mathbf{r}, A, D) = \{\mathbf{v}; \max\{|x|, |y|, |z|, |t|\} = 1, |x^2 - \mathbf{r}y^2 - ADz^2 + \mathbf{r}Dt^2| = 1/q^n\},$$

over $n \geq 0$, and of $\{\mathbf{v}; x^2 - \mathbf{r}y^2 - ADz^2 + \mathbf{r}Dt^2 = 0\}/\sim$, a set of measure zero.

Since $Y/F$ is unramified, the integral $I_s^Y(\mathbf{r}, A, D)$ is equal to

$$\sum_{n=0}^{\infty} (-1)^n q^{-nm} \operatorname{vol}(V_n^0(\mathbf{r}, A, D)).$$

The problem is then to compute the volumes

$$\operatorname{vol}(V_n^0(\mathbf{r}, A, D)) = \operatorname{vol}(V_n(\mathbf{r}, A, D))/(1 - 1/q) \qquad (n \geq 0).$$

Choose $u$ to be a non square unit. In Lemmas II.1 and II.2, $E_3 = F(\sqrt{A})$ is unramified over $F$.

**II.1. Lemma.** *When the quadratic form is* $x^2 - y^2 + \boldsymbol{\pi}(t^2 - uz^2)$ (*thus* $\mathbf{r} = 1$, $A = u$, $D = \boldsymbol{\pi}$ *up to squares*), *we have*

$$\operatorname{vol}(V_n^0) = \begin{cases} 1 - 1/q, & \text{if } n = 0, \\ 2/q - 1/q^2 + 1/q^3, & \text{if } n = 1, \\ 2q^{-n}(1 - 1/q), & \text{if } n \geq 2. \end{cases}$$

*Proof.* In our case

$$V_0 = \{(x, y, z, t); \max\{|x|, |y|, |z|, |t|\} = 1, |x^2 - y^2 + \boldsymbol{\pi}(t^2 - uz^2)| = 1\}.$$

This is the same case as that of Lemma I.1. The volume is then

$$\mathrm{vol}(V_0) = \left(1 - \frac{1}{q}\right)^2.$$

*Consider the case of* $V_n$ with $n \geq 2$. If $|x| = 1$, then put $c = c(x,t,z) = x^2 + \boldsymbol{\pi}(t^2 - uz^2)$. Since $|\boldsymbol{\pi}(t^2 - uz^2)| < 1$, we have $c(x,t,z) \in R^{\times 2}$, and we can apply Lemma I.0. Thus we obtain

$$\int_{|t|\leq 1} \int_{|z|\leq 1} \int_{|x|=1} \int_{|c-y^2|=q^{-n}} dydxdzdt = \left(1 - \frac{1}{q}\right) \frac{2}{q^n} \left(1 - \frac{1}{q}\right) = \frac{2}{q^n} \left(1 - \frac{1}{q}\right)^2.$$

If $|x| < 1$ it follows that $|y| < 1$. Thus $\max\{|z|, |t|\} = 1$. Since $t^2 - uz^2$ does not represent zero non trivially, we have $|t^2 - uz^2| = 1$, which is a contradiction. Thus

$$\mathrm{vol}(V_n) = \frac{2}{q^n} \left(1 - \frac{1}{q}\right)^2.$$

*Consider the case of* $n = 1$. Note that in this case $|x^2 - y^2| < 1$.
(A) Case of $|t^2 - uz^2| < 1$. Since $t^2 - uz^2$ does not represent zero non trivially, it follows that $|z| < 1$, $|t| < 1$. Thus $|\boldsymbol{\pi}(t^2 - uz^2)| < 1/q^2$, and

$$V_1 = \{(x,y,z,t);\, \max\{|x|,|y|\} = 1,\, |x^2 - y^2| = 1/q\}.$$

Applying Lemma I.0, the contribution of this case is

$$\int_{|z|<1} \int_{|t|<1} \int_{|x|=1} \int_{|x^2-y^2|=q^{-1}} dydxdtdz = \frac{1}{q^2} \left(1 - \frac{1}{q}\right) \frac{2}{q} \left(1 - \frac{1}{q}\right).$$

(B1) Case of $|t^2 - uz^2| = 1$ and $|x^2 - y^2| < 1/q$. Since $t^2 - uz^2$ does not represent zero non trivially, we have

$$\int \int_{|t^2-uz^2|=1} dzdt = \int_{|t|=1} \int_{|z|\leq 1} dzdt + \int_{|t|<1} \int_{|z|=1} dzdt = \left(1 - \frac{1}{q}\right) \left(1 + \frac{1}{q}\right).$$

Furthermore,

$$\int \int_{|x^2-y^2|\leq 1/q^2} dxdy = \int_{|x|=1} \int_{|x^2-y^2|\leq 1/q^2} dydx + \int_{|x|<1} \int_{|y|<1} dydx = \left(1 - \frac{1}{q}\right) \frac{2}{q^2} + \frac{1}{q^2}.$$

Thus the contribution of this case is equal to

$$\left(1 - \frac{1}{q}\right) \left(1 + \frac{1}{q}\right) \left[\left(1 - \frac{1}{q}\right) \frac{2}{q^2} + \frac{1}{q^2}\right].$$

(B2) Case of $|t^2 - uz^2| = 1$ and $|x^2 - y^2| = 1/q$. Set $w = x - y$, $v = x + y$ ($dwdv = dxdy$). Then $|wv| = 1/q$ and $|wv + \boldsymbol{\pi}(t^2 - uz^2)| = 1/q$, and the contribution of this case is (there are two integrals that correspond to $|w| = 1$, $|v| = 1/q$ and $|w| = 1/q$, $|v| = 1$):

$$2 \int \int_{|t^2-uz^2|=1} \int_{|v|=1/q} \int_{|w|=1, |wv+\pi(t^2-uz^2)|=1/q} dwdvdtdz.$$

Since $|(t^2 - uz^2)(v/\boldsymbol{\pi})^{-1}| = 1$, we have that $w \neq 0, -(t^2 - uz^2)(v/\boldsymbol{\pi})^{-1} \pmod{\boldsymbol{\pi}}$. So, the above integral is equal to

$$2\left(1 - \frac{1}{q}\right)\left(1 + \frac{1}{q}\right)\frac{1}{q}\left(1 - \frac{1}{q}\right)\left(1 - \frac{2}{q}\right).$$

Adding the contributions from Cases (A), (B1), and (B2) (divided by $(1 - 1/q)$), we obtain

$$\mathrm{vol}(V_1^0) = \frac{2}{q^3}\left(1 - \frac{1}{q}\right) + \frac{1}{q^2}\left(1 + \frac{1}{q}\right)\left(3 - \frac{2}{q}\right) + \frac{2}{q}\left(1 - \frac{1}{q^2}\right)\left(1 - \frac{2}{q}\right).$$

Once simplified this is equal to $2/q - 1/q^2 + 1/q^3$. The lemma follows. □

**II.2. Lemma.** *When the quadratic form is* $x^2 - uy^2 + u\boldsymbol{\pi}(t^2 - z^2)$ (*thus* $\mathbf{r} = u$, $A = u$, $D = \boldsymbol{\pi}$ *up to squares*), *we have*

$$\mathrm{vol}(V_n^0) = \begin{cases} 1 + 1/q, & \text{if } n = 0, \\ q^{-2}(1 - 1/q), & \text{if } n = 1, \\ 2q^{-(n+1)}(1 - 1/q), & \text{if } n \geq 2. \end{cases}$$

*Proof. Consider the case of* $n = 0$. Then

$$V_0 = \{(x, y, z, t); \max\{|x|, |y|, |z|, |t|\} = 1, |x^2 - uy^2 + u\boldsymbol{\pi}(t^2 - z^2)| = 1\}.$$

Since $x^2 - uy^2$ does not represent zero non trivially, we have

$$1 = |x^2 - uy^2 + u\boldsymbol{\pi}(t^2 - z^2)| = |x^2 - uy^2| = \max\{|x|, |y|\}.$$

We obtain

$$\mathrm{vol}(V_0) = \int_{|x|=1} \int_{|y|\leq 1} dydx + \int_{|x|<1} \int_{|y|=1} dydx = 1 - \frac{1}{q} + \frac{1}{q}\left(1 - \frac{1}{q}\right) = \left(1 - \frac{1}{q}\right)\left(1 + \frac{1}{q}\right).$$

*Consider the case of* $n = 1$. Then

$$V_1 = \{(x, y, z, t); \max\{|x|, |y|, |z|, |t|\} = 1, |x^2 - uy^2 + u\boldsymbol{\pi}(t^2 - z^2)| = 1/q\}.$$

If $\max\{|x|,|y|\}=1$, then $|x^2-uy^2|=1$. It implies that $|x|<1$, $|y|<1$, and $|x^2-uy^2|\le 1/q^2$. The contribution from this subset is equal to

$$\int_{|x|<1}\int_{|y|<1}\int\int_{|t^2-z^2|=1} dtdzdydx = \frac{1}{q^2}\left[\int_{|t|=1}\int_{|t^2-z^2|=1} dzdt + \int_{|t|<1}\int_{|z|=1} dzdt\right]$$

$$= \frac{1}{q^2}\left[\left(1-\frac{1}{q}\right)\left(1-\frac{2}{q}\right)+\frac{1}{q}\left(1-\frac{1}{q}\right)\right] = \frac{1}{q^2}\left(1-\frac{1}{q}\right)^2.$$

*Consider the case of* $n\ge 2$. Then

$$V_n = \{(x,y,z,t);\ \max\{|x|,|y|,|z|,|t|\}=1,\ |x^2-uy^2+u\boldsymbol{\pi}(t^2-z^2)|=q^{-n}\}.$$

If $\max\{|x|,|y|\}=1$ then $|x^2-uy^2|=1$, which is a contradiction. Hence $|x|<1$, $|y|<1$, and $\max\{|z|,|t|\}=1$. The latter implies that $|z|=|t|=1$. Dividing by $u\boldsymbol{\pi}$, we have

$$V_n = \{(x,y,z,t);\ |x|<1,\ |y|<1,\ |z|=1,\ |t|=1,\ |z^2-t^2+\boldsymbol{\pi}((y/\boldsymbol{\pi})^2-u^{-1}(x/\boldsymbol{\pi})^2)|=q^{1-n}\}.$$

Applying Lemma I.0 (with $c=z^2+\boldsymbol{\pi}((y/\boldsymbol{\pi})^2-u^{-1}(x/\boldsymbol{\pi})^2)$), its volume is equal to

$$\int_{|x|<1}\int_{|y|<1}\int_{|z|=1}\int_{|c-t^2|=q^{1-n}} dtdzdydx = \frac{1}{q^2}\frac{2}{q^{n-1}}\left(1-\frac{1}{q}\right)^2.$$

Dividing by $(1-1/q)$ we obtain the $\mathrm{vol}(V_n^0)$. The lemma follows. □

**II.3. Lemma.** *When the quadratic form is* $x^2-y^2+ut^2-u\boldsymbol{\pi}z^2$ *or* $x^2-y^2-uz^2+\boldsymbol{\pi}t^2$, *we have*

$$\mathrm{vol}(V_n^0) = \begin{cases} 1, & \text{if } n=0,\\ 1/q, & \text{if } n=1,\\ q^{-n}(1-1/q^2), & \text{if } n\ge 2.\end{cases}$$

*Proof.* Since the quadratic form $x^2-y^2-uz^2+\boldsymbol{\pi}t^2$ is equal to $-(y^2-x^2+uz^2-\boldsymbol{\pi}t^2)$, the computations for this form are identical to those of $x^2-y^2+ut^2-u\boldsymbol{\pi}z^2$.

*Consider the case of* $n=0$. Then

$$V_0 = \{(x,y,z,t);\ \max\{|x|,|y|,|z|,|t|\}=1,\ |x^2-y^2+ut^2-u\boldsymbol{\pi}z^2|=1\}.$$

We have the following three cases.

(A) Case of $|t|<1$. It follows that $|x^2-y^2|=1$. The contribution of this case is

$$\int_{|z|\le 1}\int_{|t|<1}\int\int_{|x^2-y^2|=1} dydxdtdz = \frac{1}{q}\left[\int_{|x|=1}\int_{|x^2-y^2|=1} dydx + \int_{|x|<1}\int_{|y|=1} dydx\right]$$

$$= \frac{1}{q}\left[\left(1-\frac{1}{q}\right)\left(1-\frac{2}{q}\right)+\frac{1}{q}\left(1-\frac{1}{q}\right)\right] = \frac{1}{q}\left(1-\frac{1}{q}\right)^2.$$

(B1) Case of $|t|=1$, $|x^2-y^2|<1$. The contribution from this case is equal to (we apply Lemma I.0):

$$\int_{|t|=1}\int_{|x|=1}\int_{|x^2-y^2|\le q^{-1}} dydxdt + \int_{|t|=1}\int_{|x|<1}\int_{|y|<1} dydxdt$$

$$= \frac{2}{q}\left(1-\frac{1}{q}\right)^2 + \frac{1}{q^2}\left(1-\frac{1}{q}\right) = \left(1-\frac{1}{q}\right)\left(\frac{2}{q}-\frac{1}{q^2}\right).$$

(B2) Case of $|t|=1$, $|x^2-y^2|=1$. Set $w=x-y$, $v=x+y$. Then $|w|=|v|=1$ and also $|ut^2w^{-1}|=1$. Thus the contribution from this case is given by the integral

$$\int_{|t|=1}\int_{|w|=1}\int_{|v|=1,|wv+ut^2|=1} dvdwdt = \left(1-\frac{1}{q}\right)^2\left(1-\frac{2}{q}\right).$$

Adding the contributions from Cases (A), (B1), and (B2) (divided by $(1-1/q)$), we obtain

$$\mathrm{vol}(V_0^0) = \frac{1}{q}\left(1-\frac{1}{q}\right)+\frac{2}{q}-\frac{1}{q^2}+\left(1-\frac{1}{q}\right)\left(1-\frac{2}{q}\right) = 1.$$

*Consider the case of* $n\ge 2$. We have the following three cases.

(A) Case of $|x|<1$. Since the quadratic form $y^2-ut^2$ does not represent zero non trivially, we have that $|y^2-ut^2|=1$ if and only if $\max\{|y|,|t|\}=1$. It implies that $|y|<1$, $|t|<1$, and thus $|x^2-y^2+ut^2-u\boldsymbol{\pi}z^2|=|\boldsymbol{\pi}z^2|=1/q$, which is a contradiction.

(B1) Case of $|x|=1$, $|t|<1$. The contribution from this case is given by the following integral (we apply Lemma I.0):

$$\int_{|t|<1}\int_{|z|\le 1}\int_{|x|=1}\int_{|(x^2+ut^2-u\boldsymbol{\pi}z^2)-y^2|=q^{-n}} dydxdzdt = \frac{1}{q}\left(1-\frac{1}{q}\right)\frac{2}{q^n}\left(1-\frac{1}{q}\right).$$

(B2) Case of $|x|=1$, $|t|=1$. Set $w=x-y$, $v=x+y$. Then we have that $|w|=|v|=1$, and from $|wv+ut^2-u\boldsymbol{\pi}z^2|=q^{-n}$, we have

$$w = u(\boldsymbol{\pi}z^2-t^2)v^{-1}+\varepsilon v^{-1}\boldsymbol{\pi}^n, \qquad dw = \frac{1}{q^n}d\varepsilon, \qquad |\varepsilon|=1.$$

The volume of this subset is given by

$$\int_{|z|\le 1}\int_{|t|=1}\int_{|v|=1}\int_{|w|=1,|vw+ut^2-u\pi z^2|=q^{-n}} dwdvdtdz$$

$$= \left(1 - \frac{1}{q}\right)^2 \int_{|\varepsilon|=1} \frac{d\varepsilon}{q^n} = \frac{1}{q^n}\left(1 - \frac{1}{q}\right)^3.$$

Adding the contributions from cases (A), (B1), and (B2) (divided by $(1-1/q)$), we obtain

$$\mathrm{vol}(V_n^0) = \frac{2}{q}\left(1 - \frac{1}{q}\right)\frac{1}{q^n} + \left(1 - \frac{1}{q}\right)^2 \frac{1}{q^n} = \left(1 - \frac{1}{q^2}\right)\frac{1}{q^n}.$$

*Consider the case of* $n = 1$. We have the following three cases.

(A) Case of $|x| < 1$. Since the quadratic form $y^2 - ut^2$ does not represent zero non trivially, we have that $|y^2 - ut^2| = 1$ if and only if $\max\{|y|, |t|\} = 1$. Hence $|y| < 1$, $|t| < 1$, and thus $|z| = 1$. The volume of this subset is equal to

$$\int_{|x|<1}\int_{|y|<1}\int_{|t|<1}\int_{|z|=1} dzdtdydx = \frac{1}{q^3}\left(1 - \frac{1}{q}\right).$$

(B1) Case of $|x| = 1$, $|t| < 1$. Applying Lemma I.0, we have

$$\int_{|t|<1}\int_{|z|\leq 1}\int_{|x|=1}\int_{|(x^2+ut^2-u\boldsymbol{\pi}z^2)-y^2|=q^{-1}} dydxdzdt = \frac{1}{q}\left(1 - \frac{1}{q}\right)\frac{2}{q}\left(1 - \frac{1}{q}\right).$$

(B2) Case of $|x| = 1$, $|t| = 1$. Set $w = x - y$, $v = x + y$. We have that $|w| = |v| = 1$, and we arrive to the same case as that of $n \geq 2$ (with $n = 1$). The contribution is

$$\frac{1}{q}\left(1 - \frac{1}{q}\right)^3.$$

Adding the contributions from cases (A), (B1), and (B2) (divided by $(1-1/q)$), we obtain

$$\mathrm{vol}(V_1^0) = \frac{1}{q^3} + \frac{2}{q^2}\left(1 - \frac{1}{q}\right) + \frac{1}{q}\left(1 - \frac{1}{q}\right)^2 = \frac{1}{q}.$$

The lemma follows. □

**II.4. Lemma.** *When the quadratic form is* $x^2 - \boldsymbol{\pi}y^2 + u\boldsymbol{\pi}(t^2 - z^2)$, *we have*

$$\mathrm{vol}(V_n^0) = \begin{cases} 1, & \text{if } n = 0, \\ 1/q, & \text{if } n = 1, \\ q^{-n}(1 - 1/q^2), & \text{if } n \geq 2. \end{cases}$$

*Proof. Consider the case of* $n = 0$. Then

$$V_0 = \{(x, y, z, t);\ \max\{|x|, |y|, |z|, |t|\} = 1,\ |x^2 - \boldsymbol{\pi}y^2 + u\boldsymbol{\pi}(z^2 - t^2)| = 1\}.$$

Obviously we have

$$\mathrm{vol}(V_0) = \int_{|x|=1} dx = 1 - \frac{1}{q}.$$

*Consider the case of* $n \geq 2$. It follows that $|x| < 1$, and dividing by $\boldsymbol{\pi}$, we have

$$V_n = \{(x,y,z,t);\ \max\{|y|,|z|,|t|\} = 1,\ |x| < 1,\ |z^2 - t^2 + uy^2 - u\boldsymbol{\pi}(x/\boldsymbol{\pi})^2| = q^{1-n}\}.$$

This case is the same as that of Lemma II.3. We have that $\mathrm{vol}(V_n^0)$ is the product of $1/q$ and the $\mathrm{vol}(V_{n-1}^0)$ of Lemma II.3, which is equal to $q^{-1}(1-1/q^2)q^{-(n-1)} = (1-1/q^2)q^{-n}$.

*Consider the case of* $n = 1$. Then

$$V_1 = \{(x,y,z,t);\ \max\{|x|,|y|,|z|,|t|\} = 1,\ |x^2 - \boldsymbol{\pi}y^2 + u\boldsymbol{\pi}(z^2 - t^2)| = 1/q\}.$$

It follows that $|x| < 1$, and dividing by $\boldsymbol{\pi}$, we have

$$V_1 = \{(x,y,z,t);\ \max\{|y|,|z|,|t|\} = 1,\ |x| < 1,\ |z^2 - t^2 + uy^2 + u\boldsymbol{\pi}(x/\boldsymbol{\pi})^2| = 1/q\}.$$

The volume of this subset is the volume of $V_0$ of Lemma II.3 multiplied by $1/q$. The lemma follows. □

**II.5. Lemma.** *When the quadratic form is* $x^2 - uy^2 - uz^2 + u\boldsymbol{\pi}t^2$, *we have*

$$\mathrm{vol}(V_n^0) = \begin{cases} 1, & \text{if } n = 0, \\ 1/q, & \text{if } n = 1, \\ q^{-n}(1 - 1/q^2), & \text{if } n \geq 2. \end{cases}$$

*Proof.* If $-1 \in R^{\times 2}$, the form is $-u(y^2 + z^2 - u^{-1}x^2 - \boldsymbol{\pi}t^2)$, and its integral has already been considered in Lemma II.3. Thus we can take $u = -1$, so the form is $x^2 + y^2 + z^2 - \boldsymbol{\pi}t^2$.

Lemma (FF) implies that equation $x_1^2 + x_2^2 + x_3^2 = 0$ in three variables, has $q^2$ solutions over $\mathbb{F}_q$.

*Consider the case of* $n = 0$. Then

$$V_0 = \{(x,y,z,t);\ \max\{|x|,|y|,|z|,|t|\} = 1,\ |x^2 + y^2 + z^2 - \boldsymbol{\pi}t^2| = 1\},$$

namely $V_0 = \{(x,y,z,t);\ |x^2 + y^2 + z^2| = 1\}$. By Theorem 6.27 of [LN], we have

$$\int\int\int_{|x^2+y^2+z^2|<1} dxdydz = \frac{1}{q}.$$

Hence

$$\mathrm{vol}(V_0) = \int_{|t|\leq 1}\int\int\int_{|x^2+y^2+z^2|=1} dxdydzdt = 1 - \int\int\int_{|x^2+y^2+z^2|<1} dxdydz = 1 - \frac{1}{q}.$$

*Consider the case of* $n \geq 2$. Any $p$-adic number $a$ such that $|a| \leq 1$ can be written (not uniquely) as a power series in $\boldsymbol{\pi}$:

$$a = \sum_{i=0}^{\infty} a_i \boldsymbol{\pi}^i = a_0 + a_1 \boldsymbol{\pi} + a_2 \boldsymbol{\pi}^2 + \ldots, \qquad a_i \in R.$$

If $|a| = 1/q^n$ we may assume that $a_0 = a_1 = \cdots = a_{n-1} = 0$ and $a_n \neq 0$. We can write

$$x = \sum_{i=0}^{\infty} x_i \boldsymbol{\pi}^i, \quad y = \sum_{i=0}^{\infty} y_i \boldsymbol{\pi}^i, \quad z = \sum_{i=0}^{\infty} z_i \boldsymbol{\pi}^i, \quad t = \sum_{i=0}^{\infty} t_i \boldsymbol{\pi}^i.$$

Their squares are

$$x^2 = \sum_{i=0}^{\infty} a_i \boldsymbol{\pi}^i, \quad y^2 = \sum_{i=0}^{\infty} b_i \boldsymbol{\pi}^i, \quad z^2 = \sum_{i=0}^{\infty} c_i \boldsymbol{\pi}^i, \quad t^2 = \sum_{i=0}^{\infty} d_i \boldsymbol{\pi}^i,$$

where

$$a_i = \sum_{j=0}^{i} x_j x_{i-j}, \quad b_i = \sum_{j=0}^{i} y_j y_{i-j}, \quad c_i = \sum_{j=0}^{i} z_j z_{i-j}, \quad d_i = \sum_{j=0}^{i} t_j t_{i-j},$$

and $x_i$, $y_i$, $z_i$, $t_i$, $a_i$, $b_i$, $c_i$, $d_i \in R$.

We have

$$x^2 + y^2 + z^2 - \boldsymbol{\pi} t^2 = \sum_{i=0}^{\infty} f_i \boldsymbol{\pi}^i, \quad f_i \in R,$$

where $f_0 = a_0 + b_0 + c_0$, $f_i = a_i + b_i + c_i - d_{i-1}$ $(i \geq 1)$. Since $|x^2 + y^2 + z^2 - \boldsymbol{\pi} t^2| = 1/q^n$ we may assume that $f_0 = f_1 = ... = f_{n-1} = 0$ and $f_n \neq 0$. Thus we obtain the relations (modulo $\boldsymbol{\pi}$)

$$a_0 + b_0 + c_0 = 0, \ a_i + b_i + c_i - d_{i-1} = 0 \, (i = 1, ..., n-1), \ a_n + b_n + c_n - d_{n-1} \neq 0.$$

If $a_0 = b_0 = c_0 = 0$, it follows that $x_0 = y_0 = z_0 = t_0 = 0$ (i.e. $|x| < 1$, $|y| < 1$, $|z| < 1$). Then $a_1 = 2x_0 x_1 = 0$, $b_1 = 2y_0 y_1 = 0$, $c_1 = 2z_0 z_1 = 0$, and thus $d_0 = a_1 + b_1 + c_1 = 0$, i.e. $|t| < 1$. This is a contradiction, since $\max\{|x|, |y|, |z|, |t|\} = 1$. Assume that $a_0 \neq 0$ (i.e. $x_0 \neq 0$). From $a_i + b_i + c_i - d_{i-1} = 0$ $(i = 1, ..., n-1)$ it follows that (since $x_0 \neq 0$)

$$x_i = (d_{i-1} - b_i - c_i - \sum_{j=1}^{i-1} x_j x_{i-j})/(2x_0), \qquad x_n \neq (d_{n-1} - b_n - c_n - \sum_{j=1}^{n-1} x_j x_{n-j})/(2x_0),$$

where in the case of $i = 1$ the sum over $j$ is empty. Thus, we have

$$\int_{|x|=1} \int_{|t| \leq 1} \int \int_{|x^2+y^2+z^2-\pi t^2| = q^{-n}} dydzdtdx = \left(\frac{1}{q}\right)^{n-1} \left(1 - \frac{1}{q}\right).$$

Applying Lemma (FF), we have

$$\int\int\int_{|x^2+y^2+z^2|<1,\,\max\{|x|,|y|,|z|\}=1} dxdydz = \frac{1}{q}\left(1-\frac{1}{q^2}\right).$$

Thus

$$\mathrm{vol}(V_n) = \frac{1}{q^{n-1}}\left(1-\frac{1}{q}\right)\times\frac{1}{q^3}\left(q^2-1\right) = \left(1-\frac{1}{q}\right)\left(1-\frac{1}{q^2}\right)\frac{1}{q^n}.$$

*Consider the case of* $n=1$. Recall that

$$V_1 = \{(x,y,z,t);\, \max\{|x|,|y|,|z|,|t|\}=1,\, |x^2+y^2+z^2-\boldsymbol{\pi}t^2| = 1/q\}.$$

We consider two cases.
(A) Case of $\max\{|x|,|y|,|z|\}<1$, i.e. $|x|<1$, $|y|<1$, $|z|<1$, and, consequently, $|t|=1$. The contribution from this case is

$$\int_{|x|<1}\int_{|y|<1}\int_{|z|<1}\int_{|t|=1} dtdzdydx = \frac{1}{q^3}\left(1-\frac{1}{q}\right).$$

(B) Case of $\max\{|x|,|y|,|z|\}=1$. This is the same as case $n\geq 2$ (with $n=1$). It contributes $(1-q^{-1})q^{-1}(1-q^{-2})$.

Adding the contributions from Cases (A) and (B) (divided by $(1-1/q)$), we obtain

$$\mathrm{vol}(V_1^0) = \frac{1}{q^3}+\frac{1}{q}\left(1-\frac{1}{q^2}\right) = \frac{1}{q}.$$

The lemma follows. □

*Proof of Theorem II.* We are now ready to complete the proof of Theorem II. Recall that we need to compute the value at $s=0$ ($m=-2$) of the product

$$\chi_Y(b_2\mathbf{s})|4\mathbf{r}D\sqrt{A}|I_s^Y(\mathbf{r},A,D)/(T^Y\phi_0)(\mathbf{v}_0).$$

Here $I_s^Y(\mathbf{r},A,D)$ is equal to the sum

$$\sum_{n=0}^{\infty}(-1)^n q^{-nm}\,\mathrm{vol}(V_n^0(\mathbf{r},D))$$

which converges for $m>-1$ by Proposition 1 or alternatively by Lemmas II.1-II.3. The value at $m=-2$ is obtained then by analytic continuation of this sum.
*Case of Lemma* II.1. We have $|4\mathbf{r}D\sqrt{A}|=1/q$, and the integral $I_s^Y(\mathbf{r},A,D)$ is equal to

$$\mathrm{vol}(V_0^0) - q^{-m}\,\mathrm{vol}(V_1^0) + \sum_{n=2}^{\infty}(-1)^n q^{-nm}\,\mathrm{vol}(V_n^0)$$

$$= 1 - \frac{1}{q} - \left(\frac{2}{q} - \frac{1}{q^2} + \frac{1}{q^3}\right)\frac{1}{q^m} + 2\left(1 - \frac{1}{q}\right) q^{-2(m+1)} \left(1 + \frac{1}{q^{m+1}}\right)^{-1}.$$

When $m = -2$, this is

$$1 - \frac{1}{q} - q^2\left(\frac{2}{q} - \frac{1}{q^2} + \frac{1}{q^3}\right) + 2\left(1 - \frac{1}{q}\right)\frac{q^2}{1+q} = \frac{-2q}{1+q}\left(1 + \frac{1}{q^2}\right).$$

Multiplying by $|4\mathbf{r}D\sqrt{A}| = 1/q$ we obtain $-2(1+1/q^2)(1+q)^{-1}$. We are done by Proposition 2.

*Case of Lemma* II.2. We have $|4\mathbf{r}D\sqrt{A}| = 1/q$, and the integral $I_s^Y(\mathbf{r}, A, D)$ is equal to

$$\mathrm{vol}(V_0^0) - q^{-m}\,\mathrm{vol}(V_1^0) + \sum_{n=2}^{\infty}(-1)^n q^{-nm}\,\mathrm{vol}(V_n^0)$$

$$= 1 + \frac{1}{q} - \frac{1}{q^2}\left(1 - \frac{1}{q}\right)\frac{1}{q^m} + \frac{2}{q}\left(1 - \frac{1}{q}\right) q^{-2(m+1)} \left(1 + \frac{1}{q^{m+1}}\right)^{-1}.$$

When $m = -2$, this is

$$1 + \frac{1}{q} - 1 + \frac{1}{q} + \frac{2}{q}\left(1 - \frac{1}{q}\right)\frac{q^2}{1+q} = \frac{2q}{1+q}\left(1 + \frac{1}{q^2}\right).$$

Multiplying by $|4\mathbf{r}D\sqrt{A}| = 1/q$ we obtain $2(1+1/q^2)(1+q)^{-1}$.

*Case of Lemmas* II.3, II.4, II.5. The integral $I_s^Y(\mathbf{r}, A, D)$ is equal to

$$\mathrm{vol}(V_0^0) - q^{-m}\,\mathrm{vol}(V_1^0) + \sum_{n=2}^{\infty}(-1)^n q^{-nm}\,\mathrm{vol}(V_n^0)$$

$$= 1 - \frac{1}{q}\frac{1}{q^m} + \left(1 - \frac{1}{q}\right)\left(1 + \frac{1}{q}\right) q^{-2(m+1)} \left(1 + \frac{1}{q^{m+1}}\right)^{-1}.$$

When $m = -2$, this is

$$1 - \frac{1}{q}q^2 + \left(1 - \frac{1}{q}\right)\left(1 + \frac{1}{q}\right)\frac{q^2}{1+q} = 0.$$

□

## Character computation for type III

For the $\theta$-conjugacy class of type III we write out the representative $g = t \cdot \mathrm{diag}(\boldsymbol{r}, \boldsymbol{r})$ as

$$\begin{pmatrix} a_1r_1 + a_2r_2A & (a_1r_2 + a_2r_1)A & (b_1r_1 + b_2r_2A)D & (b_1r_2 + b_2r_1)AD \\ a_1r_2 + a_2r_1 & a_1r_1 + a_2r_2A & (b_1r_2 + b_2r_1)D & (b_1r_1 + b_2r_2A)D \\ b_1r_1 + b_2r_2A & (b_1r_2 + b_2r_1)A & a_1r_1 + a_2r_2A & (a_1r_2 + a_2r_1)A \\ b_1r_2 + b_2r_1 & b_1r_1 + b_2r_2A & a_1r_2 + a_2r_1 & a_1r_1 + a_2r_2A \end{pmatrix}.$$

The product ${}^t\mathbf{v}gJ\mathbf{v}$ (where ${}^t\mathbf{v}=(x,y,z,t)$) is equal to

$$(b_1r_2+b_2r_1)(t^2+z^2A-y^2D-x^2AD)+2(b_1r_1+b_2r_2A)(zt-xyD),$$

where $a_1+a_2\sqrt{A}\in E_3^\times$ and $b_1+b_2\sqrt{A}\in E_3^\times$. The trace is a function of $g$, and $r=r_1+r_2\sqrt{A}$ ranges over a set of representatives in $E_3^\times$ ($E_3=F(\sqrt{A})$) for $E_3^\times/N_{E/E_3}E^\times$.

Define the quadratic form $Q=Q(x,y,z,t)$ to be

$$\frac{rb-\tau(rb)}{2\sqrt{A}}(t^2+z^2A-y^2D-x^2AD)+(rb+\tau(rb))(zt-xyD).$$

Set $I_s^Y(r,A,D)$ to be equal to

$$\int_{V^0}|Q|^{2(s-1)}\chi_Y(Q)dxdydzdt.$$

The property of the numbers $A$, $D$ and $AD$ that we need is that their square roots generate the three distinct quadratic extensions of $F$. Thus we may assume that $\{A,D,AD\}=\{u,\boldsymbol{\pi},u\boldsymbol{\pi}\}$, where $u\in R^\times-R^{\times2}$. Of course with this normalization $AD$ is no longer the product of $A$ and $D$, but its representative in the set $\{1,u,\boldsymbol{\pi},u\boldsymbol{\pi}\}$ mod $F^{\times2}$.

**III.1. Proposition.** *(i) If $D=u$ and $A=\boldsymbol{\pi}$ (or $\boldsymbol{\pi}u$) then $\sqrt{A}\notin N_{E/E_3}E^\times=A^{\mathbb{Z}}R_3^\times$.*
*(ii) If $A=u$ and $-1\in R^{\times2}$, and $D=\boldsymbol{\pi}$ (or $\boldsymbol{\pi}u$) then $\sqrt{A}\notin N_{E/E_3}E^\times=(-D)^{\mathbb{Z}}R_3^{\times2}$.*
*(iii) If $A=u=-1\notin R^{\times2}$ and $D=\boldsymbol{\pi}$ (or $\boldsymbol{\pi}u$) then there is $d\in R^\times$ with $d^2+1\in-R^{\times2}=R^\times-R^{\times2}$, hence $d+i\in R_3^\times-R_3^{\times2}$ ($i=\sqrt{A}$) and so $d+i\in E_3^\times-N_{E/E_3}E^\times$.*

*Proof.* For (iii) note that $R^\times/\{1+\boldsymbol{\pi}R\}$ is the multiplicative group of a finite field $\mathbb{F}$ of $q$ elements. There are $1+\frac{1}{2}(q-1)$ elements in each of the sets $\{1+x^2;\,x\in\mathbb{F}\}$ and $\{-y^2;\,y\in\mathbb{F}\}$. As $2(1+\frac{1}{2}(q-1))>q$, there are $x$, $y$ with $1+x^2=-y^2$. But $y\neq0$ as $-1\notin\mathbb{F}^{\times2}$. Hence there is $x$ with $1+x^2\notin\mathbb{F}^{\times2}$, and our $d$ exists. □

Since $r$ ranges over a set of representatives for $E_3^\times/N_{E/E_3}E^\times$, by Proposition III.1 we can choose $br$ to be 1 or $\sqrt{A}$ or $d+i$. Correspondingly the quadratic form takes one of the three shapes

$$t^2+z^2A-y^2D-x^2AD,\qquad zt-xyD,\qquad\text{or}\qquad t^2-z^2-y^2D+x^2D+2d(zt-xyD).$$

**III. Theorem.** *When $Y/F$ is unramified, the value of $I_s^Y(\mathbf{r},A,D)$ at $s=0$ is 0.*

*Proof.* Assume that $br=\sqrt{A}\notin N_{E/E_3}E^\times$, thus $|br\tau(br)D|=|AD|$, and the quadratic form is $t^2+z^2A-y^2D-x^2AD$. If $|A|=1/q$ or $-1$ is a square, we can replace $A$ with $-A$. The quadratic form then becomes the same as that of type I. The result of the computation is 0, see proof of Theorem I, case of anisotropic quadratic forms and we are done in this case.

If $A=-1$, $br=d+i\notin N_{E/E_3}E^\times$, the quadratic form is $t^2-z^2-y^2D+x^2D+2d(zt-xyD)$. It is equal to $X^2-uY^2-D(Z^2-uT^2)$ with $X=t+dz$, $Y=z$, $Z=y+dx$, $T=x$ and $u=d^2+1\in R^\times-R^{\times2}$. Since $|D|=1/q$ the quadratic form is anisotropic and the result of the computation is 0 by the proof of Theorem I, case of anisotropic quadratic forms.

Assume that $br=1$, thus $|br\tau(br)D|=|D|$ and the quadratic form is $zt-xyD$. Then it is $\frac{1}{4}$ times $(z+t)^2-(z-t)^2-D[(x+y)^2-(x-y)^2]$. Since $\max\{|x|,|y|,|z|,|t|\}=1$ implies $\max\{|x+y|,|x-y|,|z+t|,|z-t|\}=1$, the result of the computation is 0 by the proof of Theorem I, cases of Lemmas I.1 and I.3. The theorem follows. □

## Character computation for type IV

For the $\theta$-conjugacy class of type IV we write the representative $g = t \cdot \operatorname{diag}(\boldsymbol{r}, \boldsymbol{r})$ (where $t = h^{-1}t^*h$, $t^* = \operatorname{diag}(\alpha, \sigma\alpha, \sigma^3\alpha, \sigma^2\alpha)$) as

$$\begin{pmatrix} a_1r_1 + a_2r_2A & (a_1r_2 + a_2r_1)A & (b'_1r_1 + b'_2r_2A)D & (b'_1r_2 + b'_2r_1)AD \\ a_1r_2 + a_2r_1 & a_1r_1 + a_2r_2A & (b'_1r_2 + b'_2r_1)D & (b'_1r_1 + b'_2r_2A)D \\ b_1r_1 + b_2r_2A & (b_1r_2 + b_2r_1)A & a_1r_1 + a_2r_2A & (a_1r_2 + a_2r_1)A \\ b_1r_2 + b_2r_1 & b_1r_1 + b_2r_2A & a_1r_2 + a_2r_1 & a_1r_1 + a_2r_2A \end{pmatrix}.$$

Here $E_3 = F(\sqrt{A})$ is a quadratic extension of $F$ and $E = E_3(\sqrt{D})$ is a quadratic extension of $E_3$, thus $A \in F - F^2$ and $D = d_1 + d_2\sqrt{A} \in E_3 - E_3^2$, $d_i \in F$.

If $-1 \in F^{\times 2}$ we can and do take $D = \sqrt{A}$, where $A$ is a nonsquare unit $u$ if $E_3/E$ is unramified, or a uniformizer $\boldsymbol{\pi}$ if $E_3/F$ is ramified. If $-1 \notin F^{\times 2}$ and $E_3/F$ is ramified, once again we may and do take $A = \boldsymbol{\pi}$ and $D = \sqrt{A}$.

If $-1 \notin F^{\times 2}$ and $E_3/F$ is unramified, take $A = -1$ and note that a primitive 4th root $\zeta = i$ of 1 lies in $E_3$ (and generates it over $F$). Then $E/E_3$ is unramified, generated by $\sqrt{D}$, $D = d_1 + id_2$, and we can (and do) take $d_2 = 1$ and a unit $d_1 = d$ in $F^\times$ such that $d^2 + 1 \notin F^{\times 2}$. Then $D = d + i \notin E_3^{\times 2}$. The existence of $d$ is shown as in the proof of Proposition III.1.

Further $\alpha = a + b\sqrt{D} \in E^\times$, where $a = a_1 + a_2\sqrt{A} \in E_3^\times$, $b = b_1 + b_2\sqrt{A} \in E_3^\times$, and $r = r_1 + r_2\sqrt{A} \in E_3^\times/N_{E/E_3}E^\times$. The relation $bD = b'_1 + b'_2\sqrt{A}$ defines $b'_1 = b_1d_1 + b_2d_2A$ and $b'_2 = b_2d_1 + b_1d_2$.

Recall from Corollary 1 that we need to compute

$$\left(\frac{\nu}{\mu}\right)(\det g)\frac{\Delta(g\theta)}{\Delta_C(Ng)}\int_{V^0} |{}^t\mathbf{v}gJ\mathbf{v}|^m \chi_Y({}^t\mathbf{v}gJ\mathbf{v})d\mathbf{v}. \qquad (*)$$

Since $\det g = \alpha r \cdot \sigma(\alpha r) \cdot \sigma^3(\alpha r) \cdot \sigma^2(\alpha r)$, we have

$$\left(\frac{\nu}{\mu}\right)(\det g)\frac{\Delta(g\theta)}{\Delta_C(Ng)} = |\det g|^{(1-s)/2}\left|\frac{(\alpha r - \sigma^2(\alpha r))^2}{\alpha r\sigma^2(\alpha r)} \cdot \frac{\sigma(\alpha r - \sigma^2(\alpha r))^2}{\sigma(\alpha r)\sigma^3(\alpha r)}\right|^{1/2}$$

$$= \frac{|4brD\sigma(brD)|}{|r^2(a^2 - b^2D)\sigma(r^2(a^2 - b^2D))|^{s/2}}.$$

When $s = 0$, this is $|brD\sigma(brD)|$.

The product ${}^t\mathbf{v}gJ\mathbf{v}$ (where ${}^t\mathbf{v} = (x, y, z, t)$) is then equal to

$$(b_1r_2 + b_2r_1)(t^2 + z^2A) - (b'_1r_2 + b'_2r_1)(y^2 + x^2A) + 2(b_1r_1 + b_2r_2A)zt - 2(b'_1r_1 + b'_2r_2A)xy.$$

Since $bD = b'_1 + b'_2\sqrt{A}$, this is

$$\frac{br - \sigma(br)}{2\sqrt{A}}(t^2 + z^2A) + (br + \sigma(br))zt$$

$$-\frac{brD-\sigma(brD)}{2\sqrt{A}}(y^2+x^2A)-(brD+\sigma(brD))xy.$$

Note that $r$ ranges over a set of representatives for $E_3^\times/N_{E/E_3}E^\times$, and $b$ lies in $E_3^\times$. As $b$ is fixed, we may take $br$ to range over $E_3^\times/N_{E/E_3}E^\times$.

Further, note that $E_3/F$ is unramified if and only if $E/E_3$ is unramified. Hence $br$ can be taken to range over $\{1,\boldsymbol{\pi}\}$ if $E_3/F$ is unramified, and over $\{1,u\}$ if $E_3/F$ is ramified, where $\boldsymbol{\pi}$ is a uniformizer in $F$ and $u$ is a nonsquare unit in $F$, in these two cases. Thus in both cases we have that $\sigma(br)=br$, and the quadratic form is equal to $brQ$, where

$$Q=Q(x,y,z,t)=2zt-\frac{D-\sigma(D)}{2\sqrt{A}}(y^2+x^2A)-(D+\sigma(D))xy.$$

Thus we need to compute the value at $s=0$ of the product of $|brD\sigma(brD)|$, $\chi_Y(br)|br|^{2(s-1)}$ and

$$I_s^Y(\mathbf{r},A,D)=\int_{V^0}|Q|^{2(s-1)}\chi_Y(Q)dxdydzdt.$$

**IV. Theorem.** *When $Y/F$ is unramified, the the value of*

$$\chi_Y(br)|br|^{2(s-1)}|brD\sigma(brD)|I_s^Y(r,A,D)/(T^Y\phi_0)(\mathbf{v}_0)$$

*at $s=0$ is $-2\chi_Y(br)\delta(Y,E_3)$.*

To prove this theorem we need some auxiliary results.

**IV.1. Proposition.** *Up to a change of coordinates, the quadratic form*

$$2zt-\frac{D-\sigma(D)}{2\sqrt{A}}(y^2+x^2A)-(D+\sigma(D))xy$$

*is equal to either $x^2+\boldsymbol{\pi}y^2-2zt$ or $x^2-uy^2-2zt$ with $u\in R^\times-R^{\times 2}$. It is always isotropic.*

*Proof.* In the cases when $D=\sqrt{A}$, we have $\sigma(D)=-D$. When $D=d+i$, $\sigma D=d-i$. Thus the quadratic form takes one of the following three shapes

$$2zt-(y^2+\boldsymbol{\pi}x^2),\qquad 2zt-(y^2-ux^2),\qquad 2zt-(y^2-x^2)-2dxy.$$

For the third quadratic form we have

$$2zt-(y^2-x^2)-2dxy\;=\;(x-dy)^2-(d^2+1)y^2+2zt.$$

Recall that $u=d^2+1\in R^\times-R^{\times 2}$. After the change of variables $x'=x-dy$, followed by $x'\mapsto x$, the quadratic form is $x^2-uy^2+2zt$. Change $z\mapsto -z$ to get $x^2-uy^2-2zt$. □

Recall that $Y/F$ is unramified. Then the integral $I_s^Y(\mathbf{r},A,D)$ is equal to

$$\sum_{n=0}^{\infty}(-1)^n q^{-nm}\,\mathrm{vol}(V_n^0(\mathbf{r},A,D)).$$

The problem is then to compute the volumes

$$\mathrm{vol}(V_n^0(\mathbf{r},A,D))=\mathrm{vol}(V_n(\mathbf{r},A,D))/(1-1/q)\qquad(n\geq 0).$$

**IV.2. Lemma.** *When the quadratic form is* $x^2 - uy^2 - 2zt$*, we have*

$$\mathrm{vol}(V_n^0) = \begin{cases} 1 + 1/q^2, & \text{if } n = 0, \\ q^{-n}(1-1/q)(1+1/q^2), & \text{if } n \geq 1. \end{cases}$$

*Proof. Consider the case of* $n = 0$. Then

$$V_0 = \{(x, y, z, t);\ \max\{|x|, |y|, |z|, |t|\} = 1,\ |x^2 - uy^2 - 2zt| = 1\}.$$

(A) Case of $|x^2 - uy^2| = 1$ and $|zt| < 1$. The contribution is the product of

$$\int\int_{|x^2-uy^2|=1} dxdy = \int_{|x|=1}\int_{|y|\leq 1} dydx + \int_{|x|<1}\int_{|y|=1} dydx = \left(1 - \frac{1}{q}\right)\left(1 + \frac{1}{q}\right)$$

and

$$\int\int_{|zt|<1} dzdt = \int_{|z|<1}\int_{|t|\leq 1} dtdz + \int_{|z|=1}\int_{|t|<1} dzdt = \frac{1}{q} + \frac{1}{q}\left(1 - \frac{1}{q}\right) = \frac{1}{q}\left(2 - \frac{1}{q}\right).$$

(B) Case of $|x^2 - uy^2| < 1$ and $|zt| = 1$ (i.e. $|z| = 1$, $|t| = 1$). Since $x^2 - uy^2$ does not represent zero non trivially, the condition implies that $|x| < 1$, $|y| < 1$. Thus we obtain

$$\int_{|x|<1}\int_{|y|<1}\int_{|z|=1}\int_{|t|=1} dtdzdydx = \frac{1}{q^2}\left(1 - \frac{1}{q}\right)^2.$$

(C) Case of $|x^2 - uy^2| = 1$ and $|zt| = 1$. In this case, once $x$, $y$, and $z$ are chosen, we have that $|t| = 1$, and the condition $|x^2 - uy^2 - 2zt| = 1$ implies $t \not\equiv (x^2 - uy^2)/(2z) \pmod{\boldsymbol{\pi}}$. Since $\int_{|x^2-uy^2|=1} dxdy = 1 - q^{-2}$, we obtain

$$\int\int_{|x^2-uy^2|=1}\int_{|z|=1}\int_{|t|=1, |x^2-uy^2+zt|=1} dtdzdxdy = \left(1 - \frac{1}{q^2}\right)\left(1 - \frac{1}{q}\right)\left(1 - \frac{2}{q}\right).$$

Adding the contributions from Cases (A), (B), and (C) (divided by $(1-1/q)$), we obtain

$$\mathrm{vol}(V_0^0) = \frac{1}{q}\left(1 + \frac{1}{q}\right)\left(2 - \frac{1}{q}\right) + \frac{1}{q^2}\left(1 - \frac{1}{q}\right) + \left(1 - \frac{1}{q}\right)\left(1 + \frac{1}{q}\right)\left(1 - \frac{2}{q}\right).$$

Once simplified this is equal to $1 + 1/q^2$.

*Consider the case* $n \geq 1$. We have the following two cases.

(A) Case of $|z| = 1$. Then $x^2 - uy^2 - 2zt = \varepsilon\boldsymbol{\pi}^n$, where $|\varepsilon| = 1$, and $t = (x^2 - uy^2 - \varepsilon\boldsymbol{\pi}^n)/(2z)$. Further, $dt = q^{-n}d\varepsilon$, and the contribution from this case is

$$\int_{|x|\leq 1}\int_{|y|\leq 1}\int_{|z|=1}\int_{|\varepsilon|=1} \frac{1}{q^n} d\varepsilon dzdydx = \left(1 - \frac{1}{q}\right)\frac{1}{q^n}\int_{|\varepsilon|=1} d\varepsilon = \frac{1}{q^n}\left(1 - \frac{1}{q}\right)^2.$$

(B) Case of $|z| < 1$. If $|t| < 1$, then $\max\{|x|, |y|\} = 1$, and since $x^2 - uy^2$ does not represent zero non trivially, we have that $|x^2 - uy^2 - 2zt| = |x^2 - uy^2| = 1$, which is a contradiction, since $n \geq 1$. Hence $|t| = 1$. We have $x^2 - uy^2 - 2zt = \varepsilon\boldsymbol{\pi}^n$, where $|\varepsilon| = 1$. Further, from

$$|z| = \left|\frac{x^2 - uy^2}{2t} - \frac{\varepsilon}{2t}\boldsymbol{\pi}^n\right| = |x^2 - uy^2 - \varepsilon\boldsymbol{\pi}^n| < 1,$$

it follows that $|x^2 - uy^2| < 1$, and thus $|x| < 1$, $|y| < 1$. The contribution from this case is

$$\int_{|x|<1}\int_{|y|<1}\int_{|t|=1}\int_{|\varepsilon|=1}\frac{1}{q^n}d\varepsilon dt dy dx = \frac{1}{q^n}\left(1 - \frac{1}{q}\right)^2\frac{1}{q^2}.$$

Adding the contributions from Cases (A) and (B) (divided by $(1 - 1/q)$), we obtain

$$\mathrm{vol}(V_n^0) = \left(1 - \frac{1}{q}\right)\left(1 + \frac{1}{q^2}\right)\frac{1}{q^n}.$$

The lemma follows. □

*Proof of Theorem IV.* We are now ready to complete the proof of Theorem IV. Recall that we need to compute the value at $s = 0$ ($m = -2$) of $I_s^Y(\mathbf{r}, A, D)$. Since $Y/F$ is unramified, the integral $I_s^Y(\mathbf{r}, A, D)$ coincides with the sum

$$\sum_{n=0}^{\infty}(-1)^n q^{-nm}\,\mathrm{vol}(V_n^0(A, D))$$

which converges for $m > -1$. The value at $m = -2$ is obtained then by analytic continuation of this sum.

*Case of* $x^2 + \boldsymbol{\pi}y^2 - 2zt$. Make a change of variables $z \mapsto 2u^{-1}z'$, followed by $z' \mapsto z$. Thus the quadratic form is equal to

$$-u^{-1}((z - t)^2 - (z + t)^2 - ux^2 - u\boldsymbol{\pi}y^2).$$

Note that up to a multiple by a unit, this is a form of Lemma II.3. Since $\max\{|z|, |t|\} = 1$ implies $\max\{|z + t|, |z - t|\} = 1$, the result of that lemma holds for our quadratic form as well. In this case $E_3/F$ is ramified, and our integral is zero.

*Case of* $x^2 - uy^2 - 2zt$. This is the case where $E_3/F$ is unramified. By Lemma IV.2, the integral

$$I_s^Y(\mathbf{r}, A, D) = \mathrm{vol}(V_0^0) + \sum_{n=1}^{\infty}(-1)^n q^{-nm}\,\mathrm{vol}(V_n^0)$$

is equal to

$$1 + \frac{1}{q^2} + \left(1 - \frac{1}{q}\right)\left(1 + \frac{1}{q^2}\right)\frac{-1}{q^{(m+1)}}\left(1 - \frac{-1}{q^{m+1}}\right)^{-1}.$$

When $m = -2$, this is

$$1 + \frac{1}{q^2} + \left(1 - \frac{1}{q}\right)\left(1 + \frac{1}{q^2}\right)\frac{-q}{q + 1} = \frac{2}{1 + q}\left(1 + \frac{1}{q^2}\right).$$

The theorem follows by Proposition 2. □

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