Quasi-Controllability and Estimates of Amplitudes of Transient Regimes in Discrete Systems*

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ABSTRACT

Families of regimes for discrete control systems are studied possessing a special quasi-controllability property that is similar to the Kalman controllability property. A new approach is proposed to estimate the amplitudes of transient regimes in quasi-controllable systems. Its essence is in obtaining of constructive a priori bounds for degree of overshooting in terms of the quasi-controllability measure. The results are applicable for analysis of transients, classical absolute stability problem and, especially, for stability problem for desynchronized systems.

INTRODUCTION

Currently, there are a growing number of cases in which systems are described as operating permanently as if in the transient mode. Examples are flexible manufacturing systems, adaptive control systems with high level of external noises, so called desynchronized or asynchronous discrete event systems [1–3]. In connection with this, it is necessary to ensure that the state vector amplitude satisfies reasonable estimates within the whole time interval of the system functioning including the interval of transient regime and an infinite interval when the state vector is "close to equilibrium". Emphasize, that this necessity often contradicts the usual desire to design a feedback which makes the system as stable as possible. The reason is that the stability property characterizes only the asymptotic behavior of a system and does not take into account system behavior during the transient interval. As a result, a stable system can have large overshooting or "peaks" in the transient process that can result in complete failure of a system. First mentions about systems with peak effects could be found in [4, 5] and [6]. In [7–11] this effect was studied for some classes of linear systems. As it was noted in [9, 12–17] when the regulator in feedback links is chosen to guarantee as large degree of stability as possible then, simultaneously, overshooting of the system state during the transient process grows i.e., the peak effects are getting more dangerous.

The above papers were mainly concerned with continuous time control systems because, in completely controllable and observable discrete system it is possible to chose feedback which turns to zero the specter of the respective closed-loop system. Nevertheless, similar effects occurred when optimizing asymptotic behaviour of *badly controllable or observable* discrete time systems which arise in some applications.

Consider as the simplest, if trivial, example the linear system described by the relations

$$x_{n+1} = Ax_n + bx_n^1, \qquad n = 0, 1, 2, \ldots. \tag{1}$$

Here $x_n = \{x_n^1, x_n^2\} \in \mathbb{R}^2$, vector $b \in \mathbb{R}^2$ defines a feedback to be constructed and the matrix

$$A = \begin{pmatrix} a & \varepsilon \\ \varepsilon & a \end{pmatrix}$$

depends on a small real parameter ε.

From the asymptotical point of view the best vector b is $(-2a, -(a^2 + \varepsilon^2)/\varepsilon)$ which makes the eigenvalues of a closed system equal to zero. On the other hand, for small ε this vector b is the most dangerous at the first time step, because the closed system (1) can be written for this b as $x_{n+1} = A_* x_n$ where

$$A_* = \begin{pmatrix} -a & \varepsilon \\ a^2/\varepsilon & a \end{pmatrix}$$

has a big element a^2/ε in the left bottom corner.

There arises a general question if this kind of the peak effect is connected only with poor controllability or observability of the system? If an answer is positive, then the respective quantitative estimates are of interest. Especially urgent such estimates seems to be when a whole class of systems is examined just as in problems of absolute stability or in desynchronized systems. Another schemes of appearing peak effects in discrete systems see in [18, 19].

In this paper a new approach is developed presenting the means to solve efficiently, for some classes of systems, the problem of estimation the state vector amplitude within the whole time interval. The key concept used is a quasi-controllability property of a system that is similar to the Kalman controllability property. The degree of quasi-controllability can be characterized by a numeric value. The main result of the paper is in proving the following: *if*

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a quasi-controllable system is stable then the amplitudes of all its state trajectories starting from the unit ball are bounded by the value reciprocal of the quasi-controllable measure. Since the measure of quasi-controllability can be easily computed, this fact becomes an efficient tool for analysis of transients. It is shown also that for quasi-controllable systems the properties of stability or instability are robust with respect to small perturbation of system's parameters. Some other results in this direction were announced in [20, 21]. Note also that a concept similar to quasi-controllability from algebraic point of view was investigated in [23, 24]

QUASI-CONTROLLABLE FAMILIES OF MATRICES

The notion of *quasi-controllability* of the system will be introduced in this section. Degree of quasi-controllability will be estimated by some nonnegative value, the *quasi-controllability measure.* The basic property of quasi-controllability measure and some examples are also discussed.

Definitions and basic properties

Let $\mathfrak{A} = \{A_1, A_2, \ldots, A_M\}$ be a finite family of real $N \times N$ matrices. The family $\mathfrak{A}$ is said to be *quasi-controllable* one if no nonzero proper subspace of $\mathbb{R}^N$ is invariant for all matrices from $\mathfrak{A}$. Evidently, a family $\mathfrak{A}$ can be quasi-controllable only if $M > 1$.

Denote by $\mathfrak{A}_k$ ($k = 1, 2, \ldots$) the set of those finite products of matrices from $\mathfrak{A} \bigcup \{I\}$ which contain no more that k factors. Define $\mathfrak{A}_k(x)$, $x \in \mathbb{R}^N$ as the set of vectors Lx, with $L \in \mathfrak{A}_k$. Denote by $\mathrm{co}(W)$ and $\mathrm{span}(W)$ respectively the convex and the linear hulls of the set $W \subseteq \mathbb{R}^N$. Set also $\mathrm{absco}(W) = \mathrm{co}(W \bigcup -W)$. Let $\|\cdot\|$ be a norm in $\mathbb{R}^N$; a ball in this norm of the radius t centered at 0 denote by $\mathbf{S}(t)$.

Theorem 1 *Suppose that $p \geq N-1$. Then a family $\mathfrak{A}$ is quasi-controllable if and only if* $\mathrm{span}\{\mathfrak{A}_p(x)\} = \mathbb{R}^N$ *for each nonzero $x \in \mathbb{R}^N$.*

Proof. Let the family $\mathfrak{A}$ be quasi-controllable and $x \in \mathbb{R}^N$ be a given nonzero vector. Introduce the sets $\mathfrak{L}_0 = \mathrm{span}\{x\}$ and $\mathfrak{L}_k = \mathrm{span}\{\mathfrak{A}_k(x)\}$, $k \geq 1$. Then

$$\mathfrak{L}_0 \subseteq \mathfrak{L}_1 \subseteq \ldots \subseteq \mathfrak{L}_p \subseteq \mathbb{R}^N. \tag{2}$$

Therefore,

$$1 \leq \dim \mathfrak{L}_0 \leq \dim \mathfrak{L}_1 \leq \ldots \leq \dim \mathfrak{L}_p \leq N. \tag{3}$$

On the other hand,

$$A_i \mathfrak{L}_j \subseteq \mathfrak{L}_{j+1} \qquad (A_i \in \mathfrak{A}, \quad 0 \leq j \leq p-1). \tag{4}$$

If $\dim \mathfrak{L}_p = N$ then $\mathfrak{L}_p = \mathrm{span}\{\mathfrak{A}_p(x)\} = \mathbb{R}^N$. If $\dim \mathfrak{L}_p < N$ then by (3) and the condition $p \geq N-1$, the equality $\dim \mathfrak{L}_j = \dim \mathfrak{L}_{j+1}$ holds for some $j \in [0, p-1]$. This and (2) imply $\mathfrak{L}_j = \mathfrak{L}_{j+1}$. By the last equality and (4) the subspace $\mathfrak{L}_j$ should be invariant with respect to all matrices from $\mathfrak{A}$; due to quasi-controllability of the family $\mathfrak{A}$ this subspace coincides with $\mathbb{R}^N$. Hence, $\mathfrak{L}_j = \mathfrak{L}_{j+1} = \ldots = \mathfrak{L}_p = \mathrm{span}\{\mathfrak{A}_p(x)\} = \mathbb{R}^N$.

Now suppose that $\mathrm{span}\{\mathfrak{A}_p(x)\} = \mathbb{R}^N$, but the family $\mathfrak{A}$ is not quasi-controllable. Then there exists a nonzero proper subspace $\mathfrak{L} \subset \mathbb{R}^N$ which is invariant with respect to all matrices from $\mathfrak{A}$. Then the inclusion $\mathrm{span}\{\mathfrak{A}_p(x)\} \subseteq \mathfrak{L}$ holds for each $x \in \mathfrak{L}$. Therefore, $\mathrm{span}\{\mathfrak{A}_p(x)\} \neq \mathbb{R}^N$. This contradiction proves the quasi-controllability of the family $\mathfrak{A}$. □

The value $\sigma_p(\mathfrak{A})$ defined by

$$\sigma_p(\mathfrak{A}) = \inf_{x \in \mathbb{R}^N, \|x\|=1} \sup\{t : \mathbf{S}(t) \subseteq \mathrm{absco}(\mathfrak{A}_p(x))\}$$

is called *p-measure of quasi-controllability* of the family $\mathfrak{A}$ (with respect to the norm $\|\cdot\|$).

Theorem 2 *Let $p \geq N-1$. The family $\mathfrak{A}$ is quasi-controllable if and only if $\sigma_p(\mathfrak{A}) \neq 0$.*

Proof. Let $\sigma_p(\mathfrak{A}) \neq 0$. Then the inclusion $\mathbf{S}\{\|x\| \sigma_p(\mathfrak{A})\} \subseteq \mathrm{absco}\{\mathfrak{A}_p(x)\}$ holds for each nonzero $x \in \mathbb{R}^N$ and moreover $\mathbb{R}^N = \mathrm{span}\{\mathfrak{A}_p(x)\}$. Therefore, by Theorem 1 the family $\mathfrak{A}$ is quasi-controllable.

Suppose now that the family $\mathfrak{A}$ is quasi-controllable but $\sigma_p(\mathfrak{A}) = 0$. Then there exist $x_n \in \mathbb{R}^N$, $\|x_n\| = 1$ and $y_n \in \mathrm{absco}\{\mathfrak{A}_p(x_n)\}$ such that $y_n \to 0$ and $t y_n \notin \mathrm{absco}\{\mathfrak{A}_p(x_n)\}$ for $t > 1$. Without loss of generality we can suppose that the sequences $\{x_n\}$ and $\{y_n/\|y_n\|\}$ are convergent: $x_n \to x$, $y_n/\|y_n\| \to z$.

By Theorem 1 the linear hull of the set $\{\mathfrak{A}_p(x)\}$ coincides with $\mathbb{R}^N$. Hence, there exist matrices L_1, L_2, $\ldots$, $L_N \in \mathfrak{A}_p$ such that the vectors $L_1 x$, $L_2 x, \ldots$, $L_N x$ are linearly independent. Then the vectors $L_1 x_n$, $L_2 x_n$, $\ldots$, $L_N x_n$ are also independent for all sufficiently large n. It means that for any n there exist numbers $\theta_1^{(n)}$, $\theta_2^{(n)}$, $\ldots$, $\theta_N^{(n)}$ such that $\theta_1^{(n)} + \theta_2^{(n)} + \ldots + \theta_N^{(n)} = 1$ and the vector

$$z_n = \sum_{i=1}^{N} \theta_i^{(n)} L_i x_n \tag{5}$$

is collinear to y_n i.e., $z_n = \eta_n y_n$ ($\eta_n > 0$).

By definition, $z_n \in \mathrm{absco}\{L_1 x_n, L_2 x_n, \ldots, L_N x_n\} \subseteq \mathrm{absco}\{\mathfrak{A}_p(x_n)\}$ and $t y_n \notin \mathfrak{A}_p(x_n)$ for $t > 1$. Therefore, $\eta_n \leq 1$. This and the condition $y_n \to 0$ imply $z_n \to 0$. Without loss of generality the sequences $\{\theta_1^{(n)}\}$, $\{\theta_2^{(n)}\}$, $\ldots$, $\{\theta_N^{(n)}\}$ can be supposed to be convergent to some limits θ_1, θ_2, $\ldots$, θ_N. Now, after transition to the limit in (5), we get $\theta_1 L_1 x + \theta_2 L_2 x + \ldots + \theta_N L_N x = 0$ and $\theta_1 + \theta_2 + \ldots + \theta_N = 1$. This contradicts the linear independence of the vectors $L_1 x$, $L_2 x$, $\ldots$, $L_N x$, and the theorem is proved. □

The following theorem is useful when a family of matrices depends on a parameter.

Theorem 3 *Let $p \geq N-1$ and $N \times N$ matrices $A_1(\tau)$, $A_2(\tau)$, $\ldots$, $A_M(\tau)$ be continuous at the point 0 with respect to the real parameter τ. Suppose that the family $\mathfrak{A}(\tau) = \{A_1(\tau), A_2(\tau), \ldots, A_M(\tau)\}$ is quasi-controllable at the point $\tau = 0$. Then the system $\mathcal{A}(\tau)$ is quasi-controllable for all sufficiently small τ and the function $\sigma_p(\mathfrak{A}(\tau))$ is continuous in τ at the point $\tau = 0$.*

Examples

Let A be an $N \times N$ matrix and $b, c \in \mathbb{R}^N$. Consider the family $\mathfrak{A} = \mathfrak{A}(A, b, c)$ which consists of the matrix A and the matrix $Q = bc^T$ with entries $q_{ij} = b_i c_j$, $i, j = 1, \ldots, N$.

Example 1 *The family $\mathfrak{A}(A, b, c)$ is quasi-controllable if and only if the pair (A, b) is completely controllable by Kalman and the pair (A, c) is completely observable.*

The following example is important in the theory of asynchronous systems (see, e.g., [1, 2]). Consider an $N \times N$ scalar matrix $A = (a_{ij})$ and introduce the family $\mathfrak{P}_1(A) = \{A_1, A_2, \ldots, A_N\}$ by equalities

$$A_i = \begin{pmatrix} 1 & 0 & \ldots & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \ldots & a_{ii} & \ldots & a_{iN} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & \ldots & 1 \end{pmatrix}.$$

Let the norm in $\mathbb{R}^N$ be defined as $\|x\| = |x_1| + \ldots + |x_N|$.

Example 2 *The family $\mathfrak{P}_1(A)$ is quasi-controllable, if and only if 1 is not an eigenvalue of A and the matrix A is irreducible. If $\mathfrak{P}_1(A)$ is quasi-controllable then $\sigma_N[\mathfrak{P}_1(A)] \geq \alpha\beta^{N-1}$ where*

$$\begin{aligned} \alpha &= \frac{1}{2N} \min\{\|(A - I)x\| : \|x\| = 1\}, \\ \beta &= \frac{1}{2} \min\{|a_{ij}| : i \neq j, a_{ij} \neq 0\}. \end{aligned}$$

Consider an example influenced by discussions with E.Kaszkurewich. Let the family $\mathfrak{V}(A)$ consists of the matrices A, $D_1 A$, $D_2 A, \ldots, D_N A$. Here A is a scalar $N \times N$ matrix with the elements a_{ij} and D_i $(i = 1, 2, \ldots, N)$ are the diagonal matrices of the form

$$D_i = \mathrm{diag}\{d_{1i}, d_{2i}, \ldots, d_{ii}, \ldots, d_{Ni}\},$$

where $d_{ij} = 1$ if $i \neq j$ and $d_{ij} = -1$ if $i = j$. Such families are called the *vertex families*. Let again $\|x\| = |x_1| + \ldots + |x_N|$.

Example 3 *The vertex family is quasi-controllable if and only if 0 is not an eigenvalue of A and the matrix A is irreducible. If the family $\mathfrak{V}(A)$ is quasi-controllable then $\sigma_N[\mathfrak{V}(A)] \geq \tilde{\alpha}\tilde{\beta}^{N-1}$ where*

$$\begin{aligned} \tilde{\alpha} &= \frac{1}{N} \min\{\|Ax\| : \ \|x\| = 1\}, \\ \tilde{\beta} &= \min\{|a_{ij}| : \ i \neq j, a_{ij} \neq 0\}. \end{aligned}$$

QUASI-CONTROLLABILITY AND THE PEAK EFFECT

This section contains the main results of the paper. There will be investigated the influence of quasi-controllability on stability, instability and transient processes of dynamical systems which are generating by linear difference equations

$$x(n + 1) = A(n)x(n) \qquad (6)$$

with varying coefficients. The conceptually simple and effective method to estimate norms of solutions of difference equations uniformly for all $n = 0, 1, 2, \ldots$ on the whole interval of existence, including the initial interval of transient mode as well as the consequent infinite interval of "asymptotic behavior", will be described.

A priori estimate of overshooting measure

Let $\mathfrak{A}$ be a family of $N \times N$ matrices. The difference equation (6) is called *Lyapunov absolutely stable with respect to the family* $\mathfrak{A}$, if there exists $\mu < \infty$ such that for each sequence $A(n) \in \mathfrak{A}$ any solution $x(\cdot)$ of the respective equation satisfies the estimate

$$\sup_{n \geq 0} \|x(n)\| \leq \mu \|x(0)\|. \qquad (7)$$

The smallest μ, for which estimates (7) hold is called *overshooting measure* of the equation (6) with respect to the family $\mathfrak{A}$, and is denoted by $\chi(\mathfrak{A})$.

Theorem 4 *Let the equation* (6) *be Lyapunov absolutely stable with respect to the quasi-controllable family $\mathfrak{A}$. Then the inequality*

$$\chi(\mathfrak{A}) \leq \frac{1}{\sigma_p(\mathfrak{A})} \qquad (8)$$

holds for each $p \geq N - 1$.

This assertion is the central result of the paper. The proof is relegated to the next subsection. Now we will discuss some ways of its application. Clearly, the Lyapunov absolute stability of the equation (6) is equivalent to the Lyapunov stability of the difference inclusion

$$x(n + 1) \in F_{\mathfrak{A}} x(n). \qquad (9)$$

where the set-valued function $F_{\mathfrak{A}}$ is defined by

$$F_{\mathfrak{A}}(x) = \mathrm{co}\{Ax : \ A \in \mathfrak{A}\}.$$

Inclusions of the form (9) embrace the usual systems of the discrete absolute stability theory [22, 25, 26]. On the other hand, the Lyapunov absolute stability follows from the absolute stability of the respective system. Consequently, it is possible to combine, when estimating overshooting measure of control systems, the classical methods of absolute stability theory with Theorem 4. An example of using this approach will be presented later in discussion of applications to desynchronized systems. Now let us give only a couple of the simplest corollaries of Theorem 4.

Corollary 5 *Let the family $\mathfrak{A}$ be quasi-controllable and suppose that the only uniformly bounded solution $\ldots, x_{-n}, \ldots, x_{-2}, x_{-1}, x_0$ of inclusion (9) is the zero solution. Then for each $p \geq 1$ any solution x_n, $n = 0, 1, \ldots$ of the inclusion (9) satisfies the inequality $\|x_n\| \leq \|x_0\| \sigma_p^{-1}(\mathfrak{A})$.*

PROOF. This corollary follows from Theorem 4 and from the principle of absence of bounded solutions in absolute stability problem (see, e.g., [22]). □

Consider now the difference equation

$$x_{n+1} = Ax_n + bu_n, \qquad n = 0, 1, \ldots \qquad (10)$$

where $b \in \mathbb{R}^N$ and the scalars u_n satisfy, for fixed $\gamma > 0$ and $c \in \mathbb{R}^N$, the inequality $|u_n| \leq \gamma|\langle c, x_n\rangle|$ (here $\langle\cdot,\cdot\rangle$ denotes Euclidean inner product in $\mathbb{R}^N$). Such equations are common in control theory (see, e.g., [25]).

Corollary 6 *Let the pair (A, b) be completely controllable and the pair (A, c) be completely observable and suppose that $\max_{|\omega|=1} \gamma|\langle c, (\omega I - A)^{-1}b\rangle| < 1$. Then for each $p \geq 1$ any solution x_n, $n = 0, 1, \dots$ of the equation (10) satisfies the inequality $\|x_n\| \leq \|x_0\|\sigma_p^{-1}(\mathfrak{A}_*)$ where $\mathfrak{A}_* = \{A - \gamma\, bc^T, A + \gamma\, bc^T\}$.*

Proof. By virtue of the example 1 the class $\mathfrak{A}_*$ is quasi-controllable. So this corollary follows from Theorem 4 and the circle criteria of absolute stability, [25]. □

Proof of Theorem 4

First of all, let us establish two auxiliary assertions. Denote by $\mathfrak{R}$ the set of all finite products of matrices from $\mathfrak{A}$. Define the *length* $l(R)$ of a matrix $R \in \mathfrak{R}$ as a smallest quantity of factors $A_1, A_2, \dots, A_q \in \mathfrak{A}$ in the representation $R = A_1A_2 \dots A_q$.

Lemma 1 *Let the family $\mathfrak{A}$ be quasi-controllable and suppose that the inequalities*

$$\|Rx_*\| > \mu\frac{1}{\sigma_p(\mathfrak{A})}\|x_*\|, \qquad \mu > 1 \tag{11}$$

hold for some $x_* \in \mathbb{R}^N$ ($x_* \neq 0$), $p \geq N - 1$, $R \in \mathfrak{R}$. Then for any $x \in \mathbb{R}^N$, $x \neq 0$, there exists a matrix $R_x \in \mathfrak{R}$ such that $\|R_xx\| \geq \mu\|x\|$, $l(R_x) \leq l(R) + p$.

Proof. Fix an arbitrary $x \in \mathbb{R}^N$, $x \neq 0$. The vector $\sigma_p(\mathfrak{A})x_*$ belongs to the set absco$\left\{\mathfrak{A}_p\left(\frac{\|x_*\|}{\|x\|}x\right)\right\}$ by the definition of the quasi-controllability measure. Therefore, there exist scalars θ_1, θ_2, …, θ_Q with

$$\sum_{i=1}^{Q}|\theta_i| \leq 1, \tag{12}$$

and matrices $L_1, L_2, \dots, L_Q \in \mathfrak{A}_p$ such that

$$\sum_{i=1}^{Q}\theta_i\frac{\|x_*\|}{\|x\|}L_ix = \sigma_p(\mathfrak{A})x_*.$$

Hence,

$$\sum_{i=1}^{Q}\theta_iRL_ix = \sigma_p(\mathfrak{A})\frac{\|x\|}{\|x_*\|}Rx_*,$$

and, farther, by (11),

$$\sum_{i=1}^{Q}\|\theta_iL_iRx\| \geq \mu\|x\|.$$

But then (see (12)) there exists i, $1 \leq i \leq Q$, such that the matrix $R_x = L_iR$ satisfies $\|R_xx\| \geq \mu\|x\|$.

It remains to note that $l(R_x) \leq l(R) + p$, due to inclusion $L_i \in \mathfrak{A}_p$. So, the lemma is proved. □

The equation (6) is said to be *absolutely exponentially unstable* (with the exponent $\lambda > 1$) in the family $\mathfrak{A}$, if for some $\kappa > 0$ and for each vector $x \in \mathbb{R}^N$, $x \neq 0$, there exists a sequence $A(n) \in \mathfrak{A}$, such that the solution $x(n)$ of the corresponding equation (6) with the initial condition $x(0) = x$ satisfies the estimate

$$\|x(n)\| \geq \kappa\lambda^n\|x(0)\|, \qquad n = 0, 1, 2, \dots. \tag{13}$$

Lemma 2 *Let the family $\mathfrak{A}$ be bounded and conditions of Lemma 1 hold. Then the equation (6) is absolutely exponentially unstable in the family $\mathfrak{A}$.*

Proof. Fix an arbitrary vector $x \in \mathbb{R}^N$, $x \neq 0$, and construct an auxiliary sequence of vectors $\{z(m)\}$, $m = 0, 1, \dots$, by relations $z(0) = x$ and

$$z(m) = R_{z(m-1)}z(m-1), \qquad m = 1, 2, \dots\ .$$

Here $R_{z(m)}$ are the matrices from Lemma 1. Then by Lemma 1

$$\|z(m)\| \geq \mu^m\|z(0)\|, \qquad m = 0, 1, 2, \dots\ . \tag{14}$$

By definition, matrices $R_{z(m)}$, $m = 0, 1, \dots$, can be represented in the form

$$R_{z(m)} = A_{m,l(m)}, \dots, A_{m,2}, A_{m,1}, \qquad A_{m,j} \in \mathfrak{A},$$

where $l(m)$ is the length of $R_{z(m)}$. Denote by $\{A(n)\}$, $n = 0, 1, \dots$, the sequence of matrices

$$A_{0,1}, \dots, A_{0,l(0)}, A_{1,1}, \dots, A_{m,1}, \dots, A_{m,l(m)}, \dots\ ,$$

and consider the solution $x(n)$ of the respective equation (6), with the initial condition $x(0) = x$. Then the relations $x(q_m) = z(m)$, $m = 0, 1, \dots$, hold with $q_0 = 0$ and

$$q_m = \sum_{i=0}^{m-1}l(i), \qquad m = 1, 2, \dots.$$

Estimates (14) imply

$$\|x(q_m)\| \geq \mu^m\|x(0)\|, \qquad m = 0, 1, \dots\ . \tag{15}$$

Norms of matrices from $\mathfrak{A}$ are uniformly bounded by the conditions of the lemma and the estimates

$$q_m - q_{m-1} = l(m-1) \leq K \qquad m = 1, 2, \dots, \tag{16}$$

hold by Lemma 1. Therefore, the inequality (15), in a little bit weaker form, can be extended on the positive integers n from the interval $(q_{m-1}, q_m]$:

$$\|x(n)\| \geq \nu\mu^m\|x(0)\|, \tag{17}$$

where $\nu > 0$, $q_{m-1} < n \leq q_m$ and $m = 0, 1, \dots$.

Now, taking into account that $q_m \leq mK$ for $m = 0, 1, \dots$, we obtain by virtue of (16) that the inequalities (17) for appropriate $\kappa > 0$ $\lambda > 1$ imply the estimate (13). The lemma is proved. □

Let us return to and finish the proof of Theorem 4. Suppose that the theorem is false. Then there exists a sequence of matrices $\{A(n) \in \mathfrak{A},\ n = 0, 1, \dots\}$ and a solution $x(n)$ of the respective equation (6), such that

$$\|x(n_0)\| > \frac{1}{\sigma_p(\mathfrak{A})}\|x(0)\| \tag{18}$$

holds for some $n_0 \geq 1$, $p \geq N-1$. The inequality (18) implies

$$\|A(n_0-1)\dots A(1)A(0)x(0)\| > \frac{1}{\sigma_p(\mathfrak{A})}\|x(0)\|.$$

Hence, by Lemma 2, the equation (6) is absolutely exponentially unstable in the family $\mathfrak{A}$ and, all the more, this equation is not Lyapunov absolutely stable with respect to this family. Obtained contradiction proves the theorem. □

Application to desynchronized systems

Recently much attention was paid to the development of methods for the analysis of dynamics of multi-component systems with asynchronously interacting subsystems (see [2, 3] for further references). As examples we can mention the systems with faults in data transmission channels, multiprocessor computing and telecommunication systems, flexible manufacturing systems and so on. It turned out that under weak and natural assumptions systems of this kind possess such a strong property as robustness. In applications the robustness is often treated as reliability of a system with respect to perturbation of various nature, e.g., drift of parameters, malfunctions or noises in data transmission channels, etc.

Let us introduce basic notions of the desynchronized systems theory. Consider a linear system S consisting of N subsystems $S_1, S_2, \dots, S_N$ that interact at some discrete instances $\{T^n\}$, $-\infty < n < \infty$. The moments of interaction may be chosen according to some deterministic or stochastic law but generally they are not known in advance. Let the state of each subsystem S_i be determined within the interval $[T^n, T^{n+1})$ by a numerical value $x_i(n)$, $-\infty < n < \infty$.

Let at each moment $T^n \in \{T^k : -\infty < k < \infty\}$ only one of subsystems S_i (say, with an index $i = i(n) \in \{1, 2, \dots, N\}$) may change its state and the law of the state updating be linear:

$$x_i(n+1) = \sum_{i=1}^{N} a_{ij}x_j(n), \qquad i = i(n).$$

Consider the matrix $A = (a_{ij})$ and introduce for each $i = 1, 2, \dots, N$ an auxiliary matrix A_i (*i-mixture* of the matrix A) that is obtained from A by replacing its rows with indexes $i \neq j$ with the corresponding rows of the identity matrix I. Then the dynamics equation for the system S can be written in the following compact form:

$$x(n+1) = A_{i(n)}x(n), \qquad -\infty < n < \infty. \tag{19}$$

The system described above is referred to as *the linear desynchronized or asynchronous system.*

Theorem 7 *Let* 1 *be not an eigenvalue of A and the matrix A be irreducible. Suppose that the desynchronized system is Lyapunov absolutely stable. Then* $\chi(\mathfrak{A}) \leq \left(\alpha\beta^{N-1}\right)^{-1}$.

The proof follows immediately from Theorem 4 and Example 2. Theorem 7 is convenient to combine with different methods of establishing stability of desynchronized systems (see bibliography in [1]).

ROBUSTNESS OF INSTABILITY

In this short conclusive section we will consider another application of above methods to qualitative analysis of discrete systems.

Consider the difference equation (6), where matrices $A(n)$ belong to a family $\mathfrak{A}(\tau) = \{A_1(\tau), A_2(\tau), \dots, A_M(\tau)\}$, which depends on a real parameter τ. Then a natural question arises about dependence of properties of the equation (6) on the parameter τ.

Theorem 8 *Let the family $\mathfrak{A}(\tau)$ be quasi-controllable and continuous at $\tau = 0$. Suppose that the equation (6) is not Lyapunov absolutely stable with respect to the family $\mathfrak{A}(0)$. Then the equation (6) is also not Lyapunov absolutely stable (and, in fact, is absolutely exponentially unstable) with respect to the family $\mathfrak{A}(\tau)$ for all sufficiently small τ.*

Proof. Let the equation (6) be not Lyapunov absolutely stable with respect to the family $\mathfrak{A}(0)$. Then there exist matrices $A(n,\tau) \in \mathfrak{A}(\tau)$, $n = 0, 1, \dots$, such that at $\tau = 0$ the solution of the respective equation (6) satisfies for some $n_0 > 0$ the inequality

$$\|x(n_0)\| > \frac{1}{\sigma_{N-1}[\mathfrak{A}(0)]}\|x(0)\|.$$

Therefore

$$\|A(n_0-1,0)\dots A(0,0)x(0)\| > \frac{1}{\sigma_{N-1}[\mathfrak{A}(0)]}\|x(0)\|.$$

On the other hand, the matrices $\{A(n,\tau)\}$ are continuous at the point $\tau = 0$, so as the functions $\sigma_{N-1}[\mathfrak{A}(\tau)]$ (see Theorem 3). Hence

$$\|A(n_0-1,\tau)\dots A(0,\tau)x(0)\| > \frac{1}{\sigma_{N-1}[\mathfrak{A}(\tau)]}\|x(0)\|$$

for all sufficiently small values of τ. Then by Lemma 2 the equation (6) is absolutely exponentially unstable with respect to the class $\mathfrak{A}(\tau)$. The theorem is proved. □

In some situations the following corollary from Theorems 3, 4 and 8 is useful.

Corollary 9 *Let a quasi-controllable family of matrices $\mathfrak{A} = \{A_1, A_2, \dots, A_M\}$ be the limit of families $\mathfrak{A}_m = \{A_{1,m}, A_{2,m}, \dots, A_{M,m}\}$. Suppose that the equation (6) is Lyapunov absolutely stable with respect to the families $\mathfrak{A}_m$, $m = 1, 2, \dots$. Then this equation is Lyapunov absolutely stable with respect to the family $\mathfrak{A}$. More than that, the families $\mathfrak{A}_m$ are quasi-controllable and the measures of overshooting $\chi(\mathfrak{A}_m)$ are uniformly bounded.*

The following two examples show that the previous corollary turns to be false without the assumption about quasi-controllability of the family $\mathfrak{A}$.

Example 4 *Consider the sequence of families $\mathfrak{E}_m = \{E_m\}$, each of which consists of the single matrix*

$$E_m = \begin{pmatrix} 1-\frac{1}{m} & 1 \\ 0 & 1-\frac{1}{m} \end{pmatrix}.$$

Then the family $\mathfrak{E} = \lim_{m\to\infty}\{\mathfrak{E}_m\}$ consists of the matrix

$$E = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

and is not quasi-controllable. Therefore the respective equation (6) is not exponentially stable with respect to the family $\mathfrak{E}$, notwithstanding that this equation is exponentially stable with respect to the families $\mathfrak{E}_m$.

Example 5 *Consider the sequence of the families $\mathfrak{F}_m = \{F_m\}$, each of which consists of the single matrix*

$$F_m = \begin{pmatrix} 1 - \frac{1}{m^2} & \frac{1}{m} \\ 0 & 1 - \frac{1}{m^2} \end{pmatrix}.$$

Then the respective limiting family $\mathfrak{F}$ includes only the identity matrix I and, therefore, is not quasi-controllable. Evidently, the equation (6) is stable with respect to the family $\mathfrak{F}$, as well as with respect to the classes $\mathfrak{F}_m$, $m = 1, 2, \ldots$. On the other hand, the measures of overshooting $\chi(\mathfrak{F}_m)$ are not uniformly bounded.

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